Benasque 2005

The topological gradient and its applications

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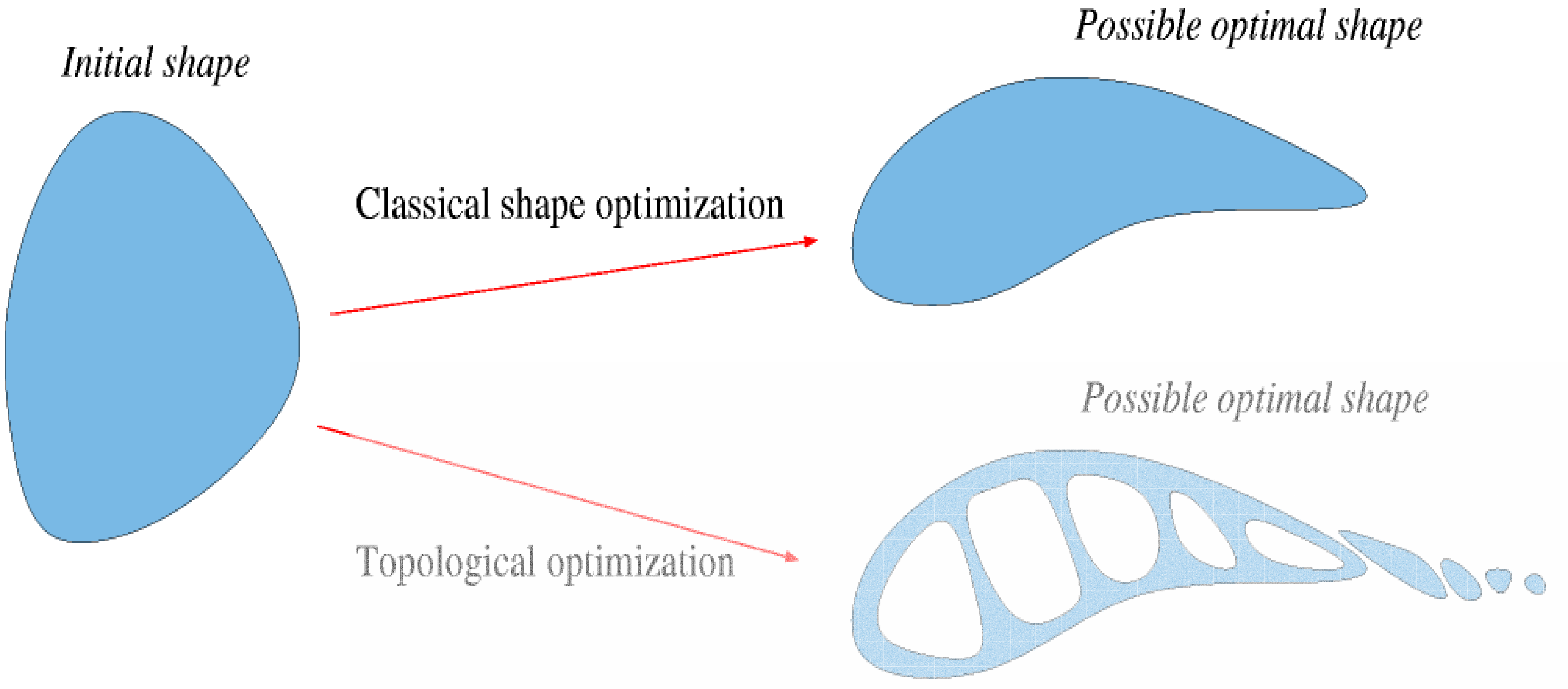
- **Introduction**
- **The topological gradient**
- **From differential calculus to 0-1 optimization**
- **A Fixed point method (J. Céa 1973)**
- **Some applications.**

Introduction

Optimal shape design: $\min_{\Omega} j(\Omega) := J(\Omega, u_{\Omega})$



Introduction

To find an optimal domain is equivalent to find its characteristic function (0-1 optimization problem).

Three ways to make this problem differentiable:

- The relaxation method: the material density function $0 \leq \theta \leq 1$ (G. Allaire, M. Bendsoe, N. Kikuchi),
- The level set method: the gradient with respect to domain variations (G. Allaire, S. Osher, F. Santosa),
- The topological asymptotic expansion: it is possible to derive the variation of a cost function if we switch from 0 to 1 or from 1 to 0 in a small area.

Level Set Approaches

In topological optimization, the unknown domain is represented by a level set function

- The relaxation method: the material density function (G. Allaire, M. Bendsoe, N. Kikuchi)

- The level set method: the built-in level set function (G. Allaire, S. Osher, F. Santosa)

- The topological asymptotic expansion: the topological gradient.
  The positivity of the topological gradient is a necessary (and even a sufficient) optimality condition.

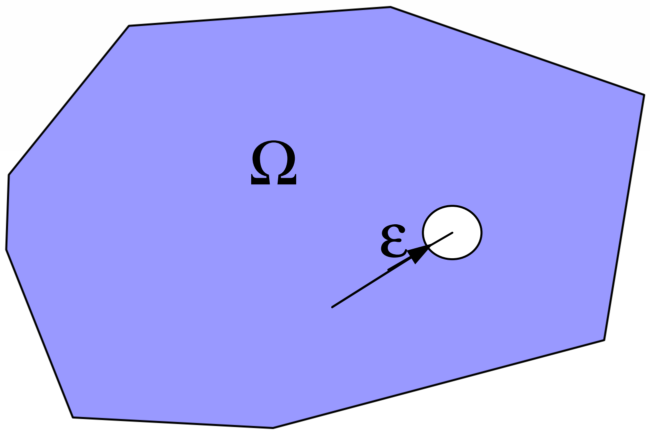
Generic form of the topological expansion

$$\Omega \mapsto u_\Omega \mapsto j(\Omega) := J(u_\Omega)$$

$$j(\Omega \backslash B(x,\varepsilon)) - j(\Omega) = f(\varepsilon)\boxed{g(x)} + o(f(\varepsilon))$$

$$f(\varepsilon) > 0 \text{ and } \lim_{\varepsilon \to 0} f(\varepsilon) = 0.$$

g **is the topological gradient**



Schumacher, J. Sokolowski, G. Allaire, Ph. Guillaume, MM

Example : the Laplace equation

		$f(\varepsilon)$	$g(x)$
Schumacher Sokolowski	Neumann 2D	$\pi\varepsilon^2$	$-2\nabla u.\nabla p$
	Neumann 3D	$\frac{4}{3}\pi\varepsilon^3$	$\frac{-3}{2}\nabla u.\nabla p$
	Dirichlet 2D	$\frac{-2\pi}{\log(\varepsilon)}$	up
	Dirichlet 3D	$4\pi\varepsilon$	up

$$Au = B \text{ and } A^*p = -\nabla J(u).$$

The expression $g = u.p$ is still valid for:

- Linear elasticity (P. Guillaume, S. Garreau, MM)
- Stokes equations (P. Guillaume, K. Sid Idriss),
- Helmholtz equation (B. Samet, S. Amstutz, MM),
- Navier-Stokes (S. Amstutz),

with a Dirichlet condition type on the boundary of the hole.

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An Example
(the linear elasticity problem)

Let us minimize the compliance

$$J(u) = B.u$$

where

$$Au = B.$$

Since $\nabla J(u) = B$, the adjoint is given by $Ap = -B$. Then $p = -u$.

Recall that $j(\Omega \backslash B(x, \varepsilon) - j(\Omega) = f(\varepsilon) u.p + \cdots$

Finally we have

$$j(\Omega \backslash B(x, \varepsilon)) - j(\Omega) = -f(\varepsilon)|u|^2 + \cdots$$

From differential calculus to 0-1 optimization

From differential calculus to 0-1 optimization

Under some hypotheses classical gradient could provide a topological asymptotic expansion.

In other cases the classical gradient provides a "topological descent direction".

Let us consider the problem

$$(\alpha A + \beta I)u = F.$$

α and β are two fonctions defined in Ω.

If α goes to 0 in $\omega \subset \Omega \Rightarrow$ a hole is created with null normal derivative of u on $\partial\omega$.

If β goes to ∞ in $\omega \subset \Omega \Rightarrow$ a hole is created with $u = 0$ on $\partial\omega$. Well known penalization method in finite element method.

General Frame

Let us consider a differentiable function

$$\begin{array}{rlcl} f: & L^p(\Omega) & \longrightarrow & \Re \\ & c & \longmapsto & f(c). \end{array} \tag{1}$$

with $1 \leq p < 2$. There exists $\gamma_1 > 0$ independent of δc such that

$$|f(c+\delta c) - f(c) - f'(c)\delta c| \leq \gamma_1 \, \|\delta c\|^2_{L^p(\Omega)} \tag{2}$$

for every δc in $L^p(\Omega)$.

Assume that

$$f'(c)\,\delta c = \int_\Omega g \;\delta c \; dx. \tag{3}$$

has a regular gradient g.

The perturbation

$$\delta c_\varepsilon = \begin{cases} 1 & \text{in} \qquad\qquad B(x_0, \varepsilon) \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

is small in $L^p(\Omega)$ for small ε :

$$||\delta c_\varepsilon||_{L^p(\Omega)} = meas(B(x_0, \varepsilon))^{1/p}.$$

Let us denote $\rho(\varepsilon) = meas(B(x_0, \varepsilon))$. The rest behaves like $||\delta c_\varepsilon||^2_{L^p(\Omega)} = (\rho(\varepsilon)^{1/p})^2 = \rho(\varepsilon)^{2/p} = o(\rho(\varepsilon))$ since $p < 2$.

The derivative $\int_\Omega g \; \delta c \; dx$ behaves like $\rho(\varepsilon)$ since g is regular : $|\int_\Omega g \; \delta c_\varepsilon \; dx - g(x_0)\rho(\varepsilon)| < \gamma_2 \varepsilon \rho(\varepsilon)$.

Let us set $j(\varepsilon) = f(c + \delta c_\varepsilon)$. The classical gradient g is the topological gradient !

$$|j(\varepsilon) - j(0) - \rho(\varepsilon) g(x_0)| = o(\rho(\varepsilon)). \tag{5}$$

The Dirichlet condition

Let us consider $\Omega \subset \Re^N, N \leq 3$, $\mathcal{V} \subset H^1(\Omega)$, the bilinear form $a(u,v)$ satisfying classical hypotheses and

$$a(c,u,v) = a(u,v) + \int_\Omega cuv\,dx.$$

We know that $H^1(\Omega) \subset L^q$ for $1 \geq \frac{1}{q} > \frac{1}{2} - \frac{1}{N}$:

$$\begin{array}{lcl} N = 2 & : & 1 \leq q < \infty \\ N = 3 & : & 1 \leq q < 6 \end{array}$$

For $q > 2$,   consider p satisfying $\frac{1}{p} = 1 - \frac{2}{q}$. The map

$$\begin{array}{ccc} L^p & \to & \mathcal{L}_2(\mathcal{V}) \\ c & \mapsto & a(c,.,.) = a(u,v) + \int_\Omega cuv\,dx \end{array}$$

is continuous.

For $N = 2$ we have $p > 1$. Then $1 < p < 2$.

For $N = 3$, we have

$$q < 6 \Rightarrow \frac{2}{q} > \frac{1}{3} \Rightarrow 1 - \frac{2}{q} < 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow p > \frac{3}{2}.$$

Then $\frac{3}{2} < p < 2$.

We consider u_c, the solution to

$$a(c, u_c, v) = l(v) \quad \forall v \in \mathcal{V}.$$

and $f(c) = J(u_c)$. Then

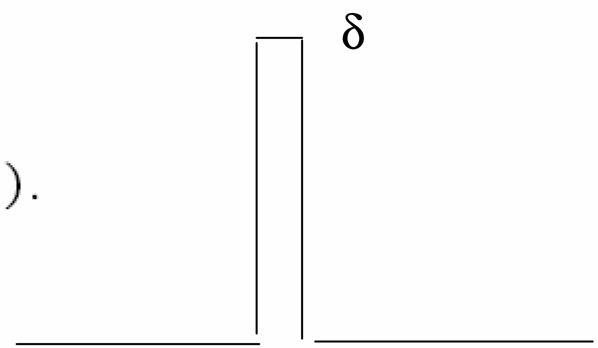
$$f'(c)\delta c = \int_\Omega \delta c \, up \, dx.$$

We have $g = up$. The regularity of g depends on a, l and J.

Let us denote $u_\varepsilon = u_{0+\delta c_\varepsilon}$ and $j(\varepsilon) = J(u_\varepsilon)$. Then

$$j(\varepsilon) - j(0) = \rho(\varepsilon)\, \delta\, up + o(\rho(\varepsilon)).$$

(Connection with the Dirichlet Problem).



2 : Application to a simple example.

Let us consider the problem

$$\begin{cases} -u'' + cu & = & 0 \quad \text{in }]0,1[\\ u(0) & = & 0 \\ u'(1) & = & 1 \end{cases} \tag{6}$$

and the cost function $J(u_c) = u_c(1)$ where u_c is the solution to (6) for a given c.

The Lagrange operator is

$$L(c,u,p) = u(1) + \int_0^1 (u'p' + cup)\,dx - p(1).$$

We have to minimize

$$\begin{array}{rlcl} f: & L^1(0,1) & \rightarrow & \Re \\ & c & \mapsto & f(c) = J(u_c) \end{array} . \tag{7}$$

In this simple case, the adjoint is $p_c = -u_c$ and

$$f'(c)\delta c = \partial_c L(c, u_c, p_c).\delta c = -\int_0^1 u_c^2\,\delta c \;\; dx.$$

We recall that $f'(c)\delta c = -\int_0^1 u_c^2\, \delta c\ dx$ and

$$\delta c_\varepsilon = \begin{cases} 1 & \text{in} \qquad\qquad [x_0, x_0 + \varepsilon] \\ 0 & \text{otherwise} \end{cases} .$$

Then

$$|j(\varepsilon) - j(0) - (-u_c(x_0)^2)\varepsilon| = o(\rho(\varepsilon)).$$

3 : Check

The exact solution for $c = 0$ is $u_0 = x$ then

$$|j(\varepsilon) - j(0) - (-x_0{}^2)\varepsilon| = o(\rho(\varepsilon)). \tag{8}$$

By calculating explicitly the solution we have

$$u_{c_\varepsilon}(1) = \frac{e^{\varepsilon}(x_0 + 1) + e^{-\varepsilon}(x_0 - 1)}{e^{\varepsilon}(x_0 + 1) - e^{-\varepsilon}(x_0 - 1)} + 1 - (x_0 + \varepsilon). \tag{9}$$

Then $j(\varepsilon) - j(0) = -x_0^2\varepsilon + \cdots$ we obtain the same result as in (8).

When the differentiability of f is limited to L^∞ (or more generally to L^p, $p \geq 2$) this result is not still valid.

Example

Let us consider the problem

$$\left\{ \begin{array}{lcl} (cu')' & = & 0 \quad \text{in }]0,1[\\ u(0) & = & 0 \\ c(1)u'(1) & = & 1 \end{array} \right. \tag{10}$$

and the cost function $J(u_c) = u_c(1)$ where u_c is the solution to (10)

The Lagrange operator is

$$L(c, u, p) = u(1) + \int_0^1 (cu'p')\, dx - p(1).$$

We have to minimize

$$\begin{array}{llll} f: & L^\infty(0,1) & \rightarrow & \Re \\ & c & \mapsto & f(c) = J(u_c). \end{array} \qquad (11)$$

In this simple case, the adjoint is $p_c = -u_c$ and

$$f'(c)\delta c = \partial_c L(c, u_c, p_c).\delta c = -\int_0^1 (u_c')^2\, \delta c \ \ dx.$$

We consider

$$\delta c_\varepsilon = \begin{cases} \delta & \text{in} \qquad\qquad [x_0, x_0+\varepsilon] \\ 0 & \text{elsewhere.} \end{cases}$$

The exact solution if $c = c_0 + \delta\varepsilon$ where c_0 is a constant is given by

$$u_{c_\varepsilon}(1) = \frac{1-\varepsilon}{c_0} + \frac{\varepsilon}{c_0+\delta} \tag{12}$$

then $j(\varepsilon) - j(0) = \varepsilon(\frac{1}{c_0+\delta} - \frac{1}{c_0})$.

Recall that

$$j(\varepsilon) - j(0) = \varepsilon(\frac{1}{c_0 + \delta} - \frac{1}{c_0}). \tag{13}$$

If we consider the gradient $f'(c)\delta c$ we obtain

$$-\varepsilon(u_c')^2\delta = -\varepsilon\frac{\delta}{c_0^2}. \tag{14}$$

The result obtained (14) is not correct if we compare it to (13). If δ is small these expressions are close. It isn't the case if δ is large.

But the two expressions have the same sign.

The Fixed point method of J. CEA (1973)

initialization $\Omega_0 \subset D$ is given

repeat for k=0, 1, . . .

1. compute u_k, p_k the direct and the adjoint solutions in the domain Ω_k,
2. compute the topological gradient g_k,
3. compute $\tilde{g}_k$ a regular extension of g_k to the domain D,
4. the new domain is given by $\Omega_{k+1} = \{x \in D; \tilde{g}_k \geq \rho_k\}$. The step size ρ_k is such that $j(\Omega_{k+1}) < j(\Omega_k)$.

This alogorithm recalls the gradient method.

SHELL ECO-MARATHON

To drive as far as possible using a given quantity of oil.

Sophie JAN: co-author and Pilote

MIP - Toulouse

Energy control and environmental protection $\Rightarrow$

SHELL eco-marathon

"To drive as far as possible using the less amount of energy"

Principle of the competition on the Nogaro motor circuit

- seven laps (D = 3.636 km per lap)
- in less than T = 50'34" (30 km/hour)

The Nogaro motor circuit

$$\min_u \left(\left(7D - x(T)\right)^+ \right)^2 + \int_0^T \text{consumption}(x(t), v(t), u(t))dt$$

$$\left\{ \begin{array}{l} x' = v \\ v' = Bv^2 + C(x, v) + D(v)u \end{array} \right\} \begin{array}{l} \text{Dynamic of the vehicle} \\ \text{(explicit Euler scheme)} \end{array}$$

$$\left\{ \begin{array}{l} x(0) = 0 \\ v(0) = 0 \end{array} \right\} \text{Initial conditions}$$

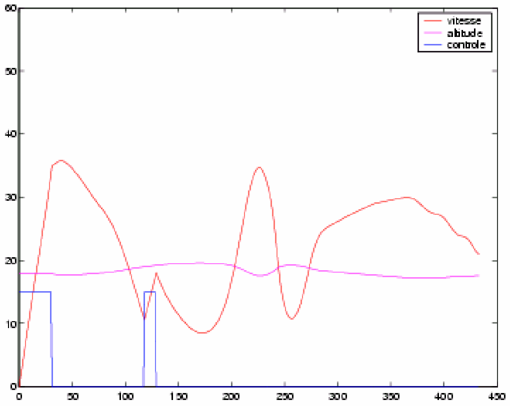
x : distance covered by the vehicle

v : speed of the vehicle

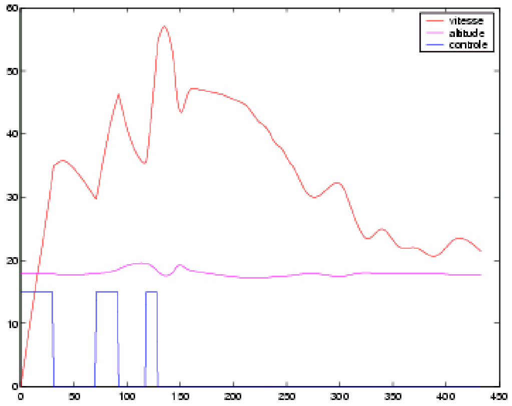
u : state of the engine $(1 = \text{on}/\text{off} = 0)$

For only one lap

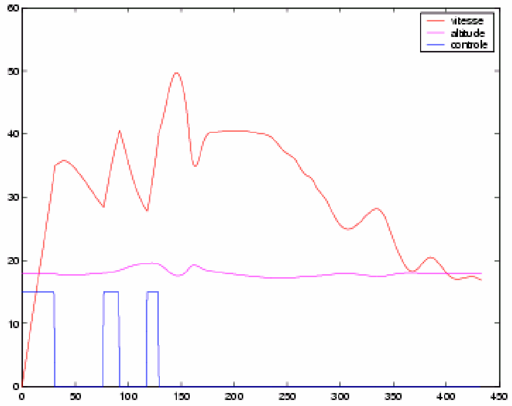
Before optimization



First iteration



Second iteration



$2720.38\text{m} < 3636.0\text{m}$ $\quad$ $4010.68\text{m} \gg 3636.0\text{m}$ $\quad$ $3671.64\text{m} \approx 3636.0\text{m}$

Attempt is lost !!!

$J = 4345.3$ $\quad$ $J = 11634.9$ $\quad$ $J = 9019.9$

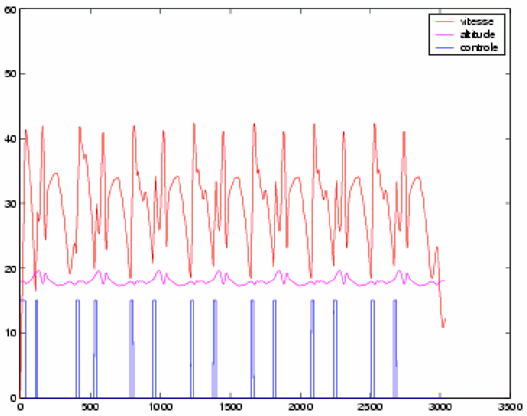
blue : state of the engine

pink : position on the circuit (via the altitude)

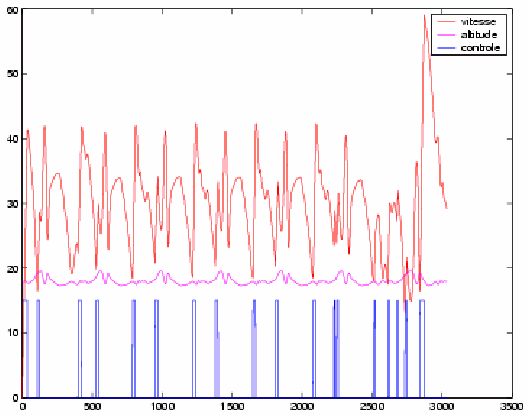
red : speed of the vehicle

For seven laps

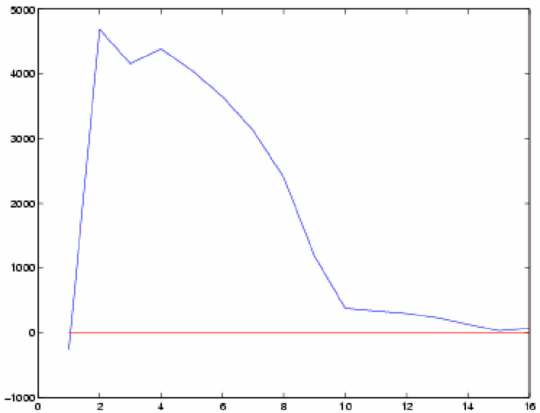
Before optimization



After optimization

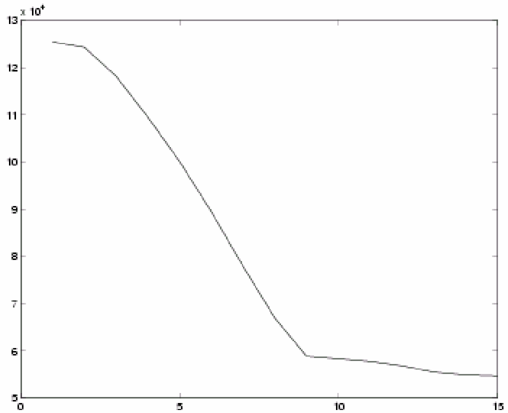


x(T)-7D



iterations

evolution of the cost



iterations

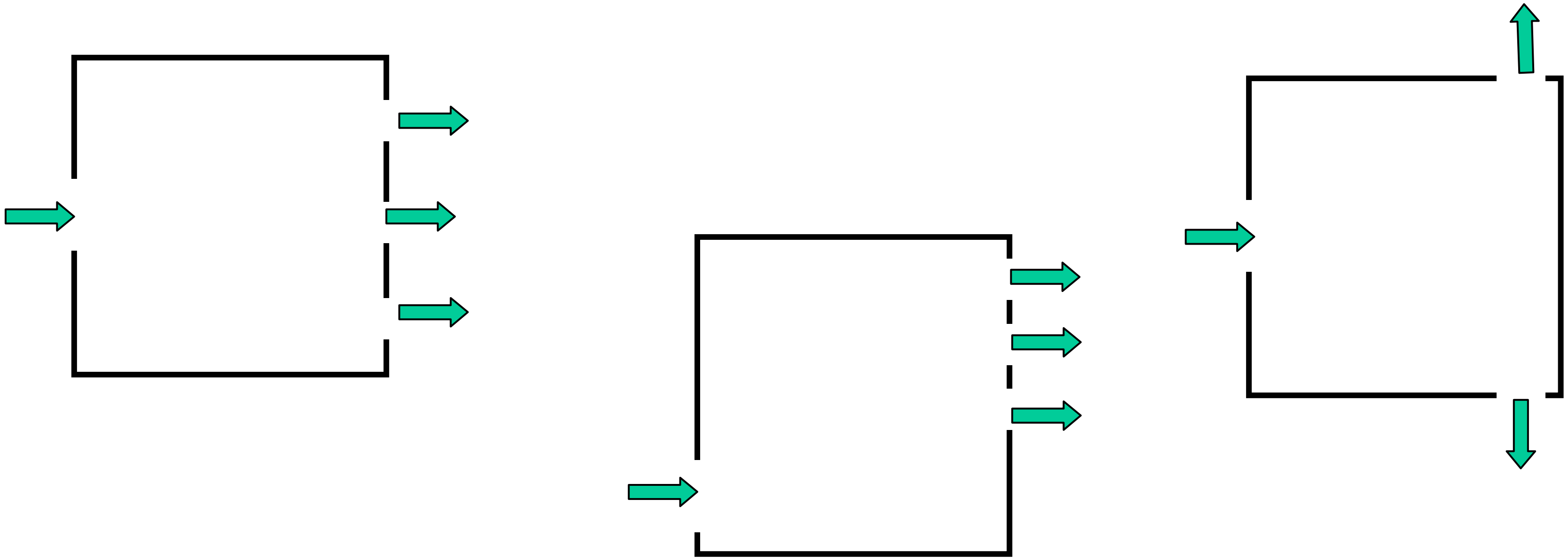
Application to Wave guides Design

Alcatel Space

Some applications to CFD problems

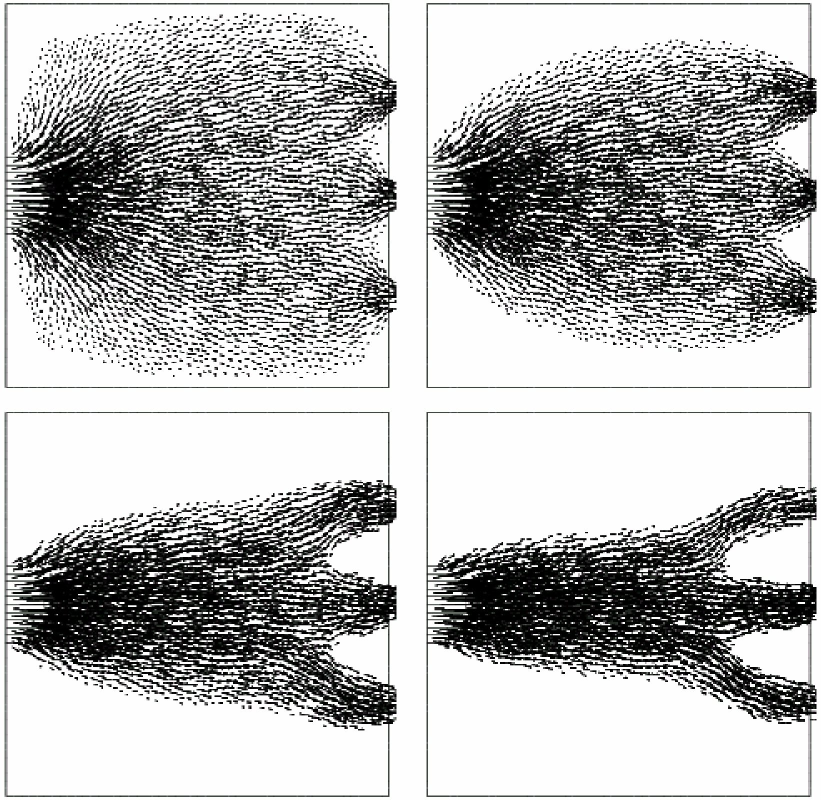
To maximize the flow for a given inlet pressure (Stokes)

with H. Maatoug (Enit)



We consider 3 cases

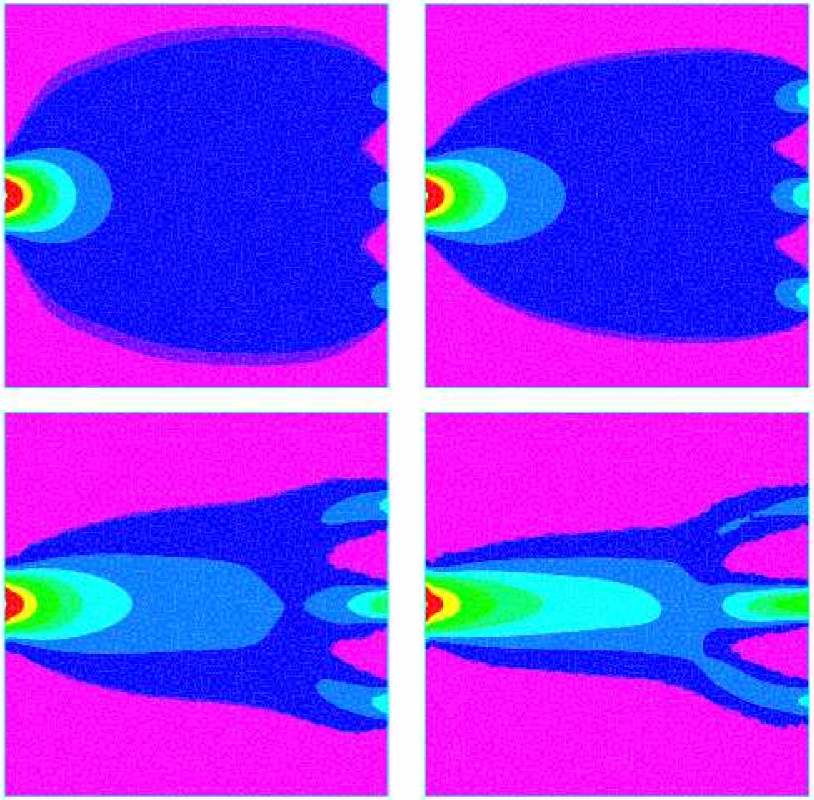
To maximize the flow, for a given inlet pressure (Stokes)



Champs de vitesse: initiale et iteration 1, 2 et 3

1

The first test case

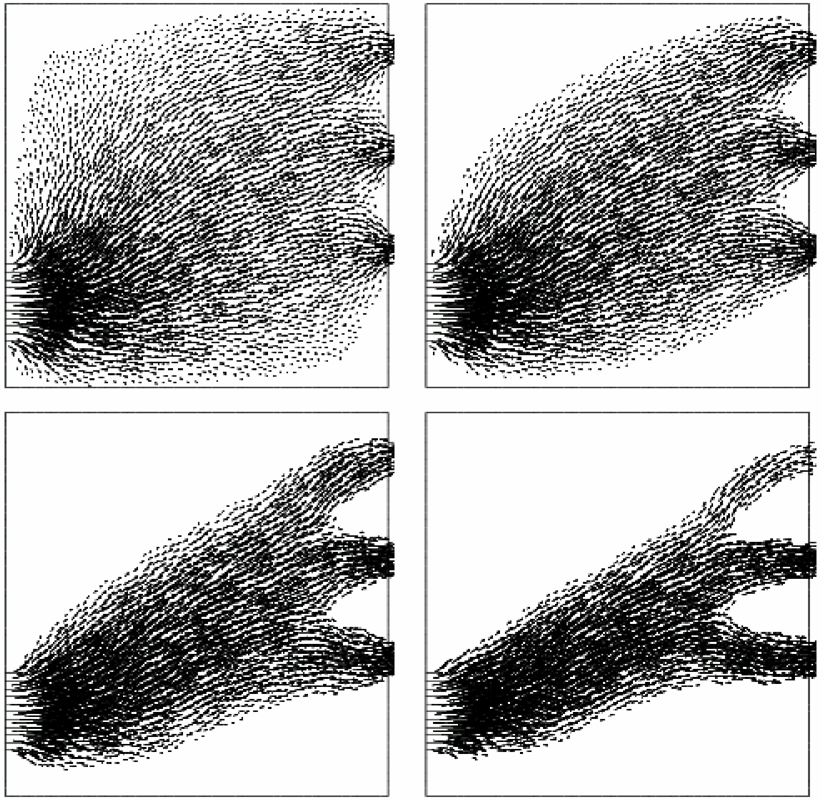


Gradient topologique: iteration 1, 2, 3 et 4

3

The topological gradient

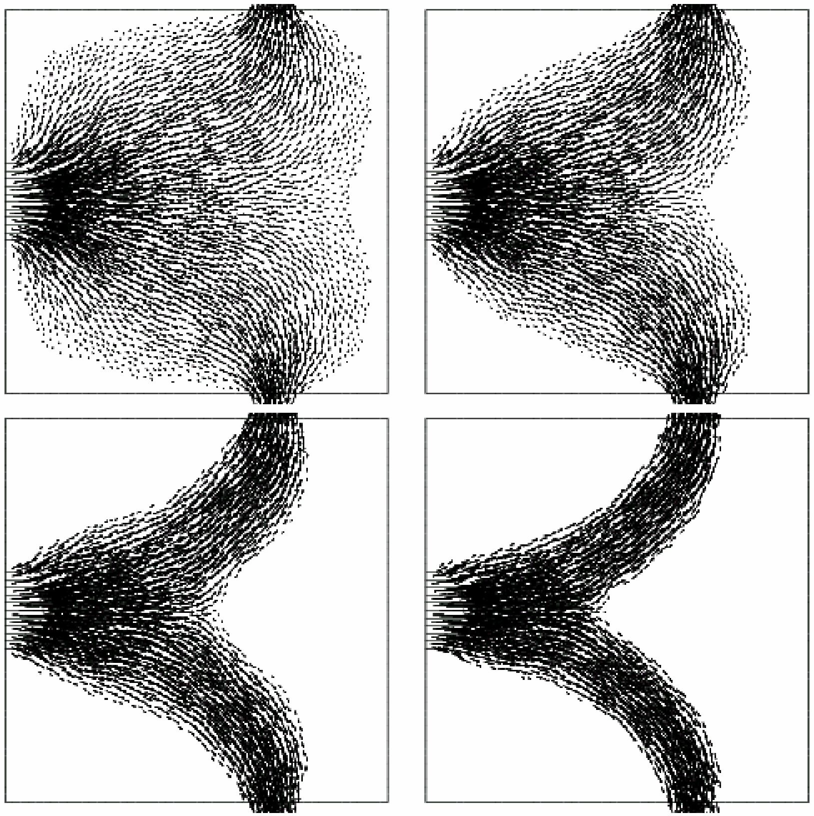
To maximize the flow, for a given inlet pressure (Stokes)



Champs de vitesse: initiale et iteration 1, 2 et 3

The second test case

To maximize the flow, for a given inlet pressure (Stokes)



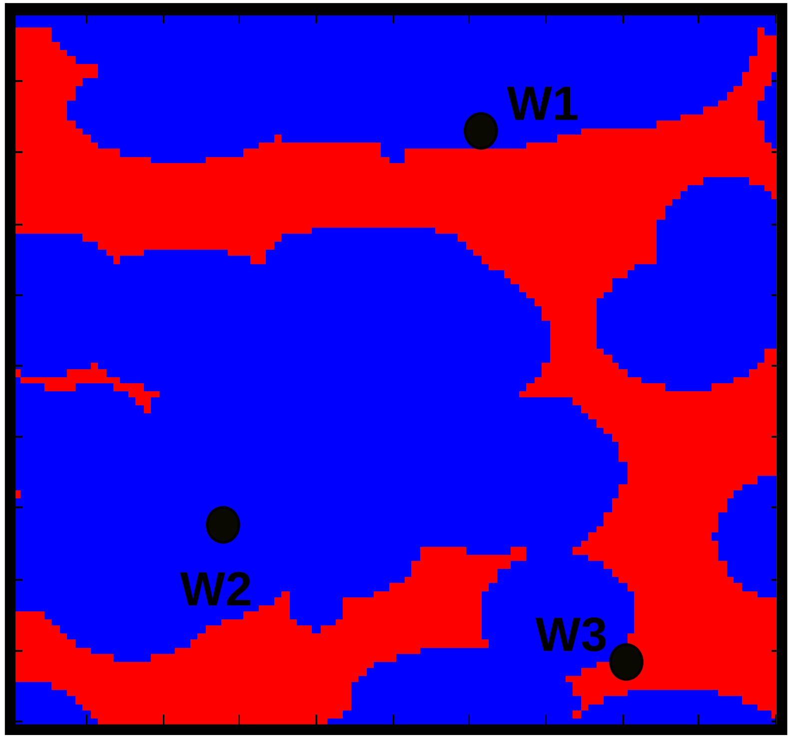
Champs de vitesse: initiale et iteration 1, 2 et 3

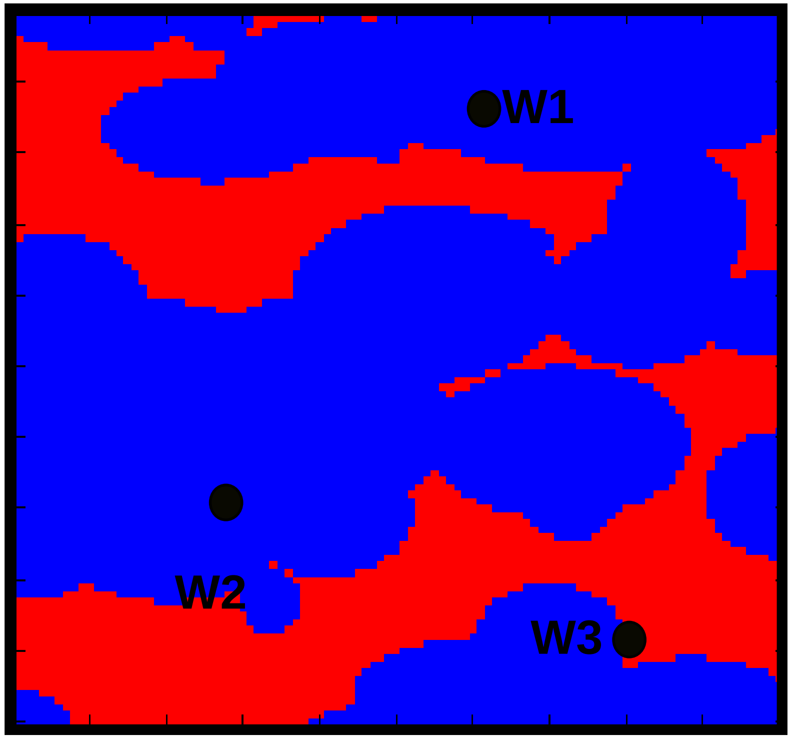
The third test case

Application to a transient nonlinear problem

Application to history matching in petrophysics
P.E. Edoa MIP – D. Rahon IFP - MM

Transient Multiphase Darcy Equations





Inverse problem example

- **Well test data :**
  pressure and derivative P_{obs} and $\frac{\partial P_{obs}}{\partial t}$

- **Inverse problem :**

$$\begin{cases} Find\ a\ domain\ \Omega\ such\ as \\ P_{\Omega}^{sim} = P^{obs}\ and\ \frac{\partial P_{\Omega}^{sim}}{\partial t} = \frac{\partial P^{obs}}{\partial t} \end{cases}$$

- **Objective function :**

$$j(\Omega) = \frac{1}{2}\int_0^T \alpha \left\| P_{\Omega}^{sim} - P^{obs} \right\|^2 + \beta \left\| \frac{\partial P_{\Omega}^{sim}}{\partial t} - \frac{\partial P^{obs}}{\partial t} \right\|^2 dt$$

- **Objective : find the domain Ω which minimizes the objective function *j***

Calculation of the topological gradient

- **Single phase flow**

$$G(x)=\int_0^T\left(c\left(\phi_{B(x,\varepsilon)}-\phi_{\Omega_\varepsilon}\right)\frac{\partial p_{\Omega_\varepsilon}}{\partial t}\lambda_{\Omega_\varepsilon}-\frac{1}{\mu}\left(K_{\Omega_\varepsilon}\nabla p_{\Omega_\varepsilon}\right).\nabla\lambda_{\Omega_\varepsilon}\right)(x,t)dt$$

p_Ω : direct state, λ_Ω : adjoint state

- **Two-phase flow**

$$G(x)=\int_0^T\left(F(\Omega_\varepsilon)-F(B(x,\varepsilon))+(K_{\Omega_\varepsilon}\nabla P_{\Omega_\varepsilon}.(\frac{kr_w}{\mu_w}\nabla u_{\Omega_\varepsilon}+(\frac{kr_w}{\mu_w}+\frac{kr_o}{\mu_o})\nabla v_{\Omega_\varepsilon}))\right)(x,t)dt$$

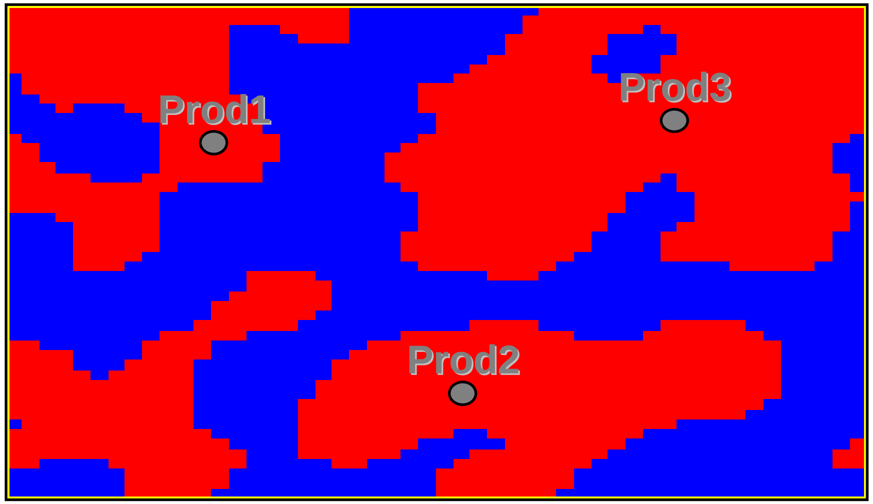
with $F(r)=(A\frac{\partial S_r}{\partial t}+B\frac{\partial(P_r S_r)}{\partial t})u_r+B\frac{\partial P_r}{\partial t}v_r$

P_Ω, S_Ω : direct state, u_Ω, v_Ω : adjoint state

Example 1 : 2D geostatistical case

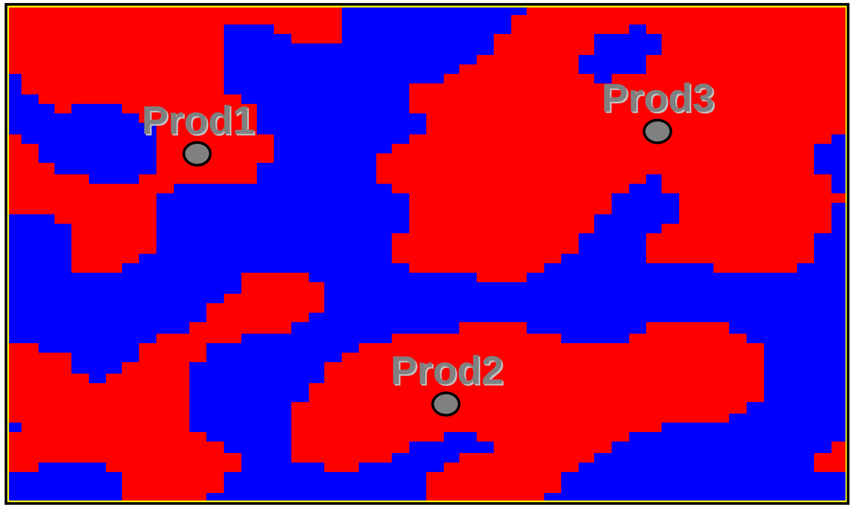
- Objective : to find images satisfying dynamic data
  - 2 facies : K : 300mD (facies 1) ,1mD (facies 2)
  - Synthetics well tests

- Simulation data
  - H = 10m, F = 0.25
  - L = 1000m, l = 500m
  - 2500 grid blocks
  - Well tests : 1 day
  - 3 wells, rate : 10m3/d

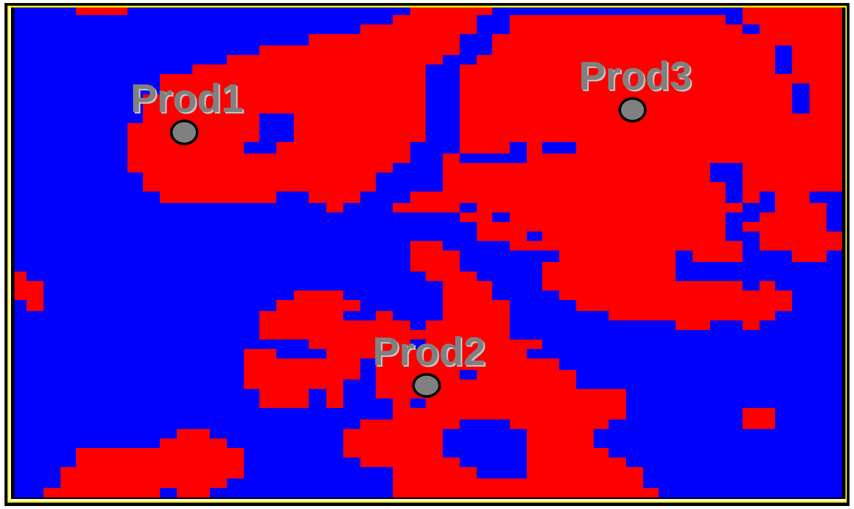


Reference permeability map

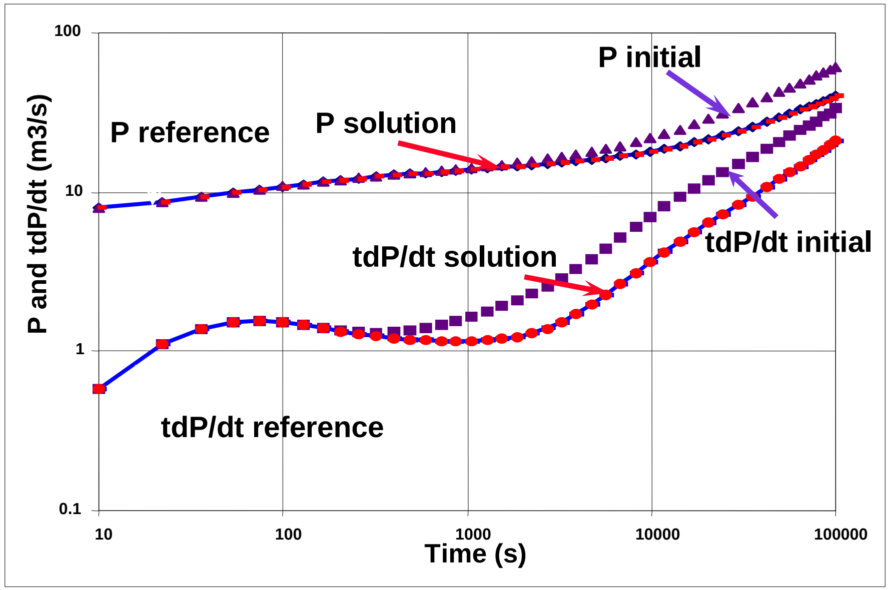
Optimization results

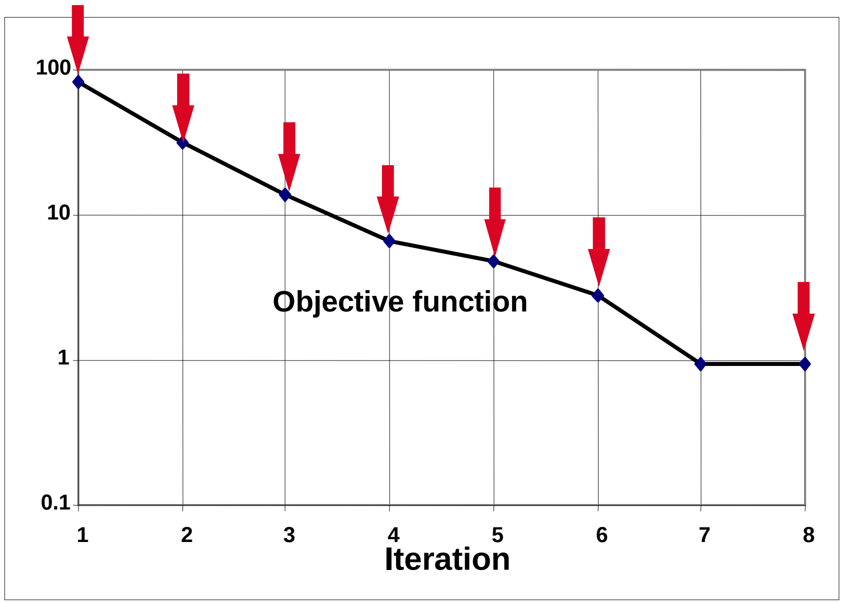


Reference permeability map



Evolution of the permeability map

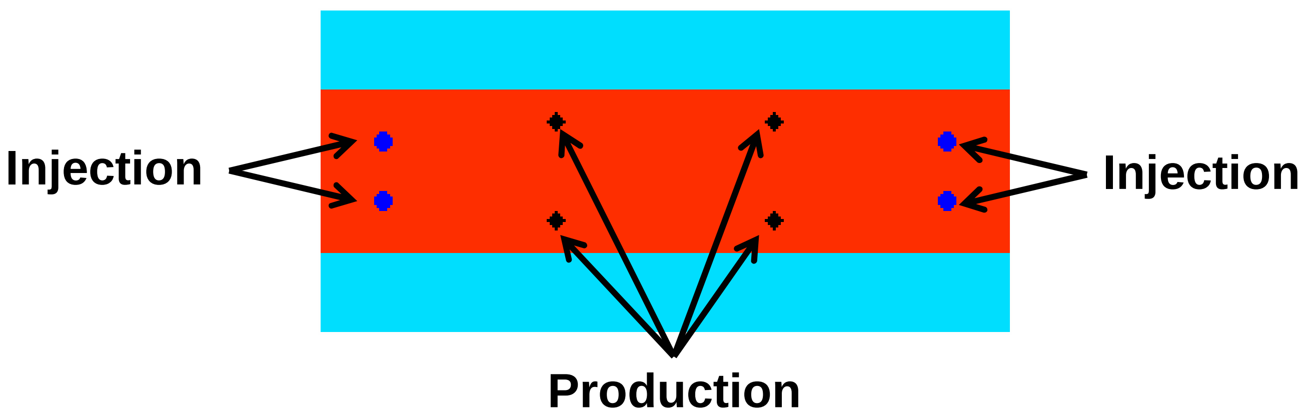




Synthetic 3D case: two-phase flow

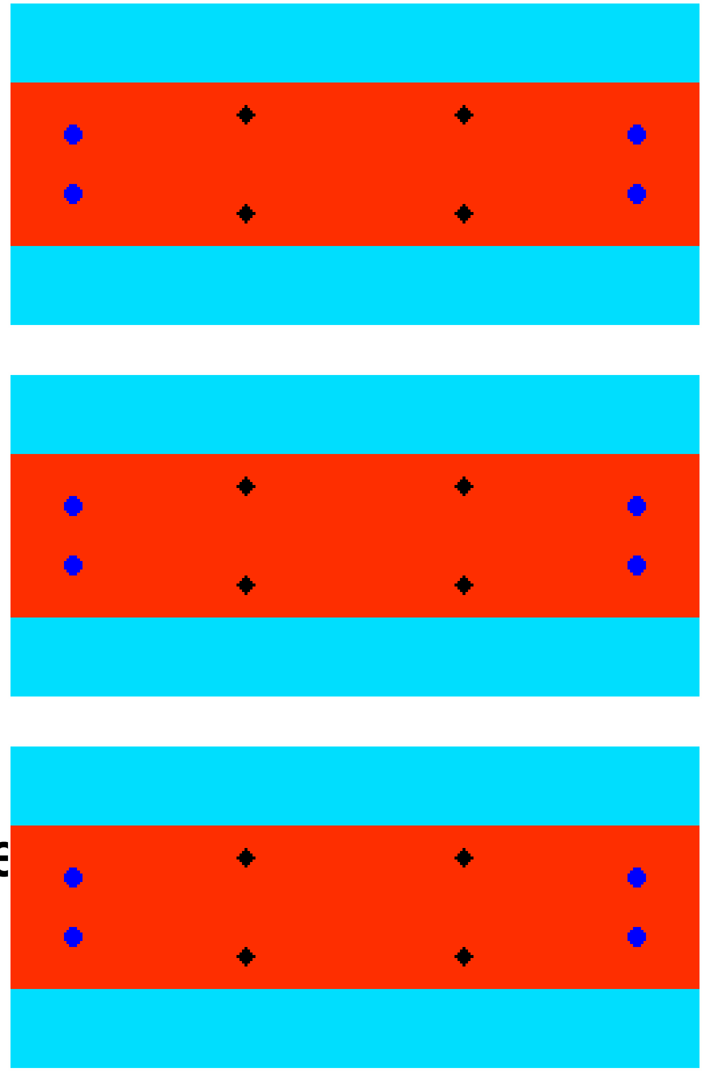
- Production data
  - 4 production wells, rate: 300m3/d (WO)
  - 4 water injection wells, rate: 100m3/d
  - Production history: 3 years
  - Vertical production wells perforated in each layers
  - Injection wells perforated in the bottom layer

Well locations



Second example: 3D, two-phase flow

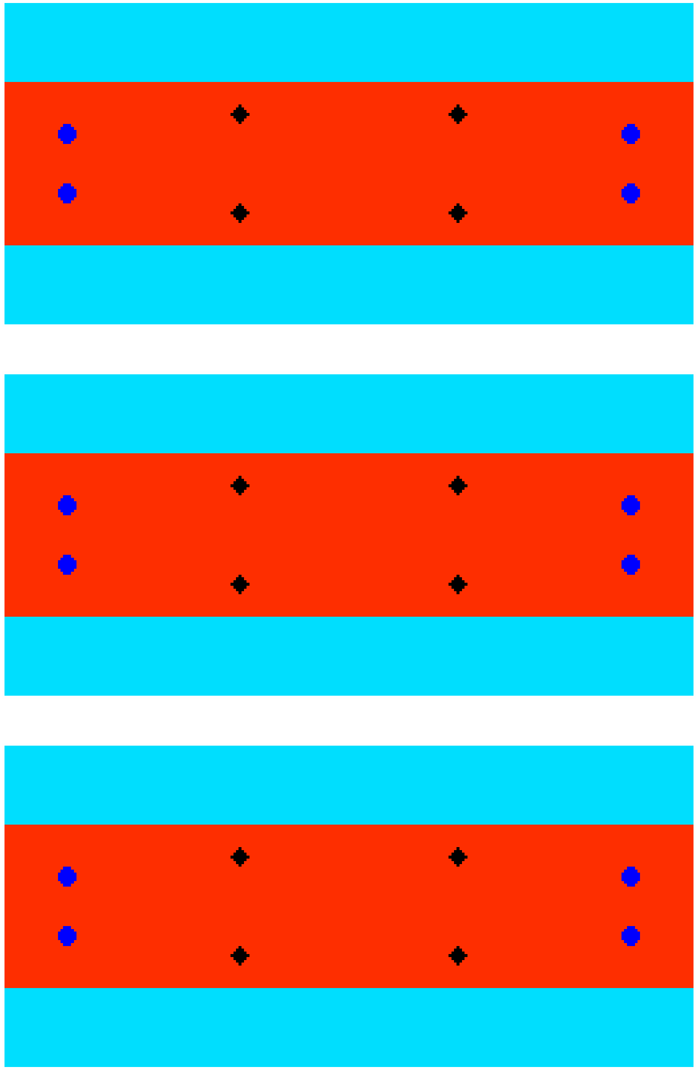
- **Objective** : **determine the reservoir volume by history matching production data (water rate and pressure)**



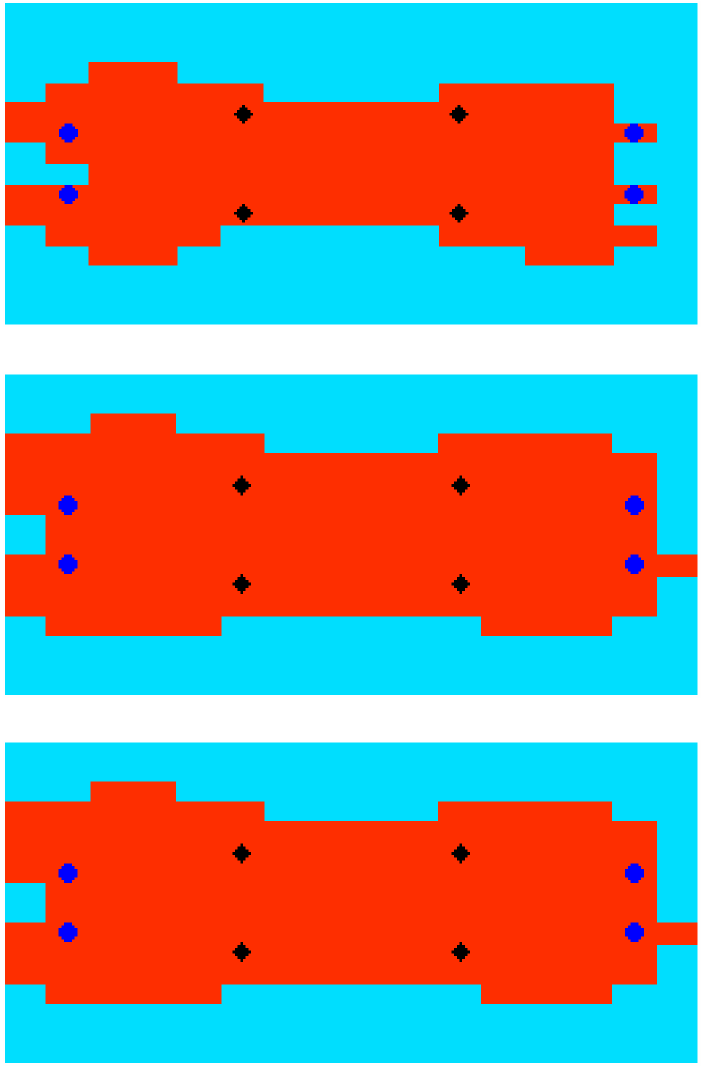
Permeability map of the 3 layers

- **Simulation model**
  - **2 facies :**
    Reservoir $K_h = 300mD$, $K_v = 10mD$
    Non reservoir $K_h = 1mD$, $K_v = 0.1mD$
  - **3 layers, 256 grid blocks per laye**
  - **L = 800m, l = 800m, H = 15m**
  - **Two-phase flow : water and oil**

Optimization results

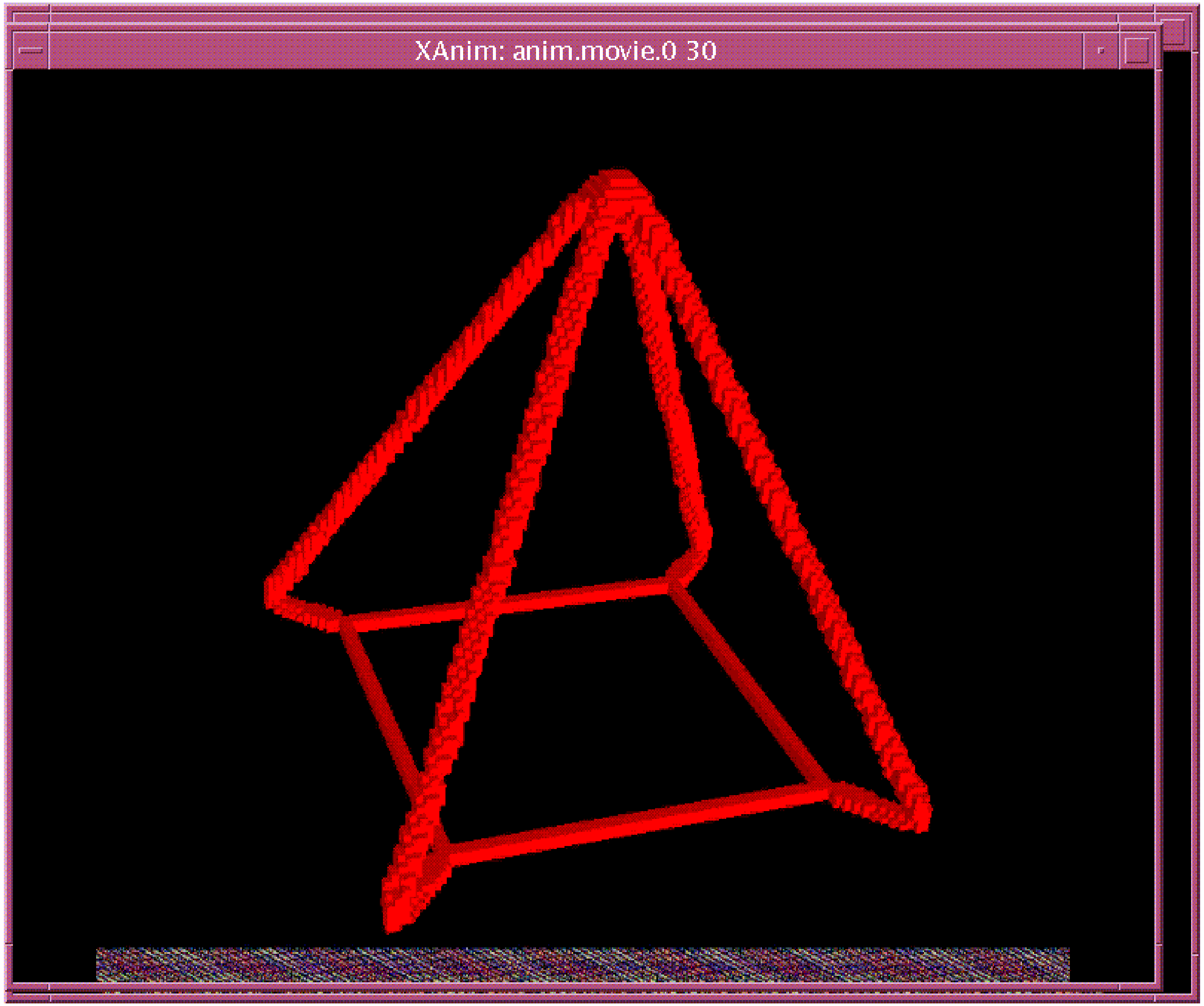


Reference permeability map of the 3 layers



Evolution of the permeability map

Structural Engineering Application



Some open questions

- The topological asymptotic expansion for
  - Transient problems
  - Steady compressible N.S.E.
- What happens if δ goes to infinity ?