Asymptotic behavior of nonautonomous reaction-diffusion equations

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The problem

We study nonautonomous parabolic equations of the form

$$\begin{cases} u_t - \Delta u = f(t,x,u) \qquad \text{in } \Omega, \quad t > s \\ u = 0, \qquad \text{on } \Gamma = \partial\Omega \\ u(s) = u_0 \geq 0. \end{cases} \tag{1}$$

- $f : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$, Typically **logistic nonlinearity:**

$$f(t,x,u) = m(t,x)u - n(t,x)u^{\rho}$$

with $n(t,x) \geq 0$, $\rho > 1$.

- the **initial data** is in $C(\overline{\Omega})$
- solutions $u_f(t,s,x,u_0)$ defined for all $t > s$.

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Part I

Some approaches to asymptotic behavior

The autonomous case: $f = f(x, u)$

- Semigroup: $X \ni u_0 \mapsto u_f(t, u_0) = S(t)u_0 \in X$
- Smoothing and estimates: compactness

$$\{S(t)u_0, \quad t \geq 1\} \quad \text{is compact}$$

- ω–limit sets:

$$\omega(u_0) = \{v_0 \in X, \ \exists \ t_n \to \infty, \ S(t_n)u_0 \to v_0\}$$

If $B \subset X$

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$$S(t)\mathcal{A} = \mathcal{A}, \quad t \geq 0$$

$$\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{A}) = 0$$

[Hale, Temam, Ladyzhenskaya .....]

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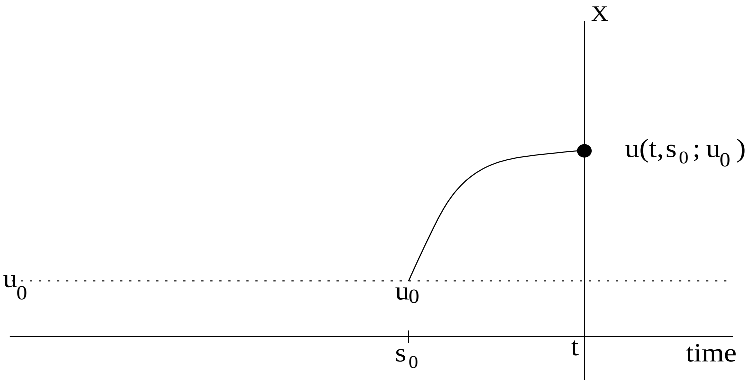
The general case: pullback attraction

For a given time t and a state u_0

the evolution that started some time before

The state ω pullback attracts the state u_0: $\lim u_F(t, s, \cdot; u_0) = \omega$

The general case: pullback attraction



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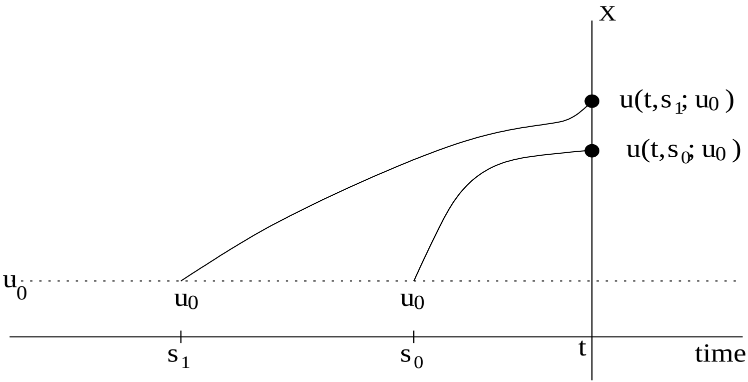

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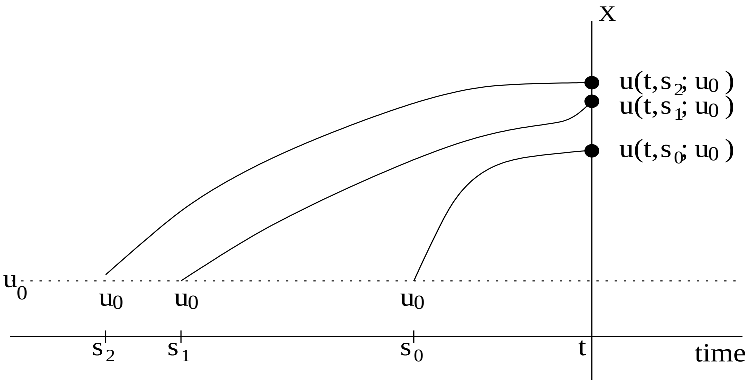

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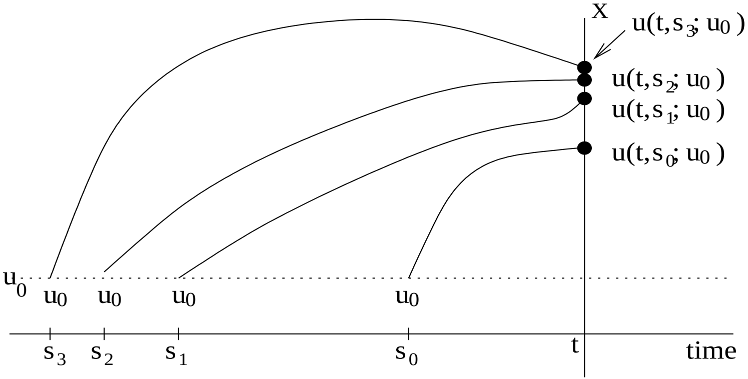

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The general case: pullback attraction

The pullback attractor is a family

$$\mathcal{A} = \{\mathcal{A}(t)\}_{t \in R}$$

such that for each $t \in I\!R$ and $B \subset X$ bounded

$$\lim_{s \to -\infty} \mathrm{dist}(u(t,s;B), \mathcal{A}(t)) = 0$$

[Robinson,Vidal-López,R-B, (2005)]

Theorem *Assume f satisfies*

$$u\ f(t,x,u) \leq C(t,x)u^2 + D(t,x)|u| \quad \forall u \in \mathbb{R}$$

and $U_{\Delta+C}(t,s)$ is exponentially stable. Definition
Then there exist $\varphi_m(t) \leq \varphi_M(t)$ minimal and maximal complete trajectories, such that

1. *Any complete trajectory ψ satisfies*

$$\varphi_m(t) \leq \psi(t) \leq \varphi_M(t);$$

2.

$$\varphi_m(t) \leq \liminf_{s\to-\infty} u(t,s;v_0) \leq \limsup_{s\to-\infty} u(t,s;v_0) \leq \varphi_M(t)$$

uniformly in Ω for $v_0 \in B \subset X$;

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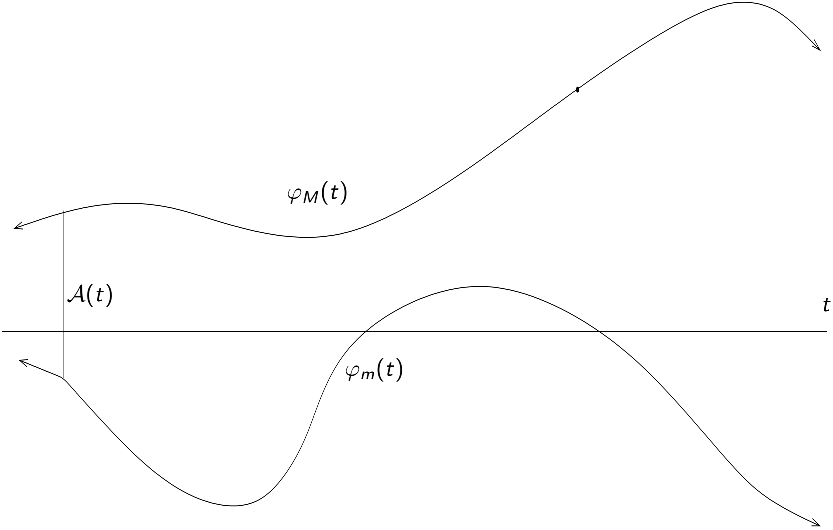
Moreover

3. *$\varphi_m(t)$ is stable from below in the pullback sense and $\varphi_M(t)$ is stable from above*
4. *there exists a pullback attractor and*

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)], \quad \varphi_m(t), \varphi_M(t) \in \mathcal{A}(t);$$

5. *the interval $[\varphi_m(t), \varphi_M(t)]$ is positively invariant*


$\varphi_M(t)$
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t
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Part II

Positive solutions

The autonomous case: $f = f(x, u)$

Assume $f(x, u)$ continuous in $u \geq 0$

Theorem (Brezis, Oswald)

$$\frac{f(x, u)}{u} \quad \textit{is } \textbf{nonincreasing} \textit{ in } u \geq 0.$$

Then there exists **at most** *a positive solution.*

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Assume for $0 \leq u \leq \delta$, $\quad f(x,u) \geq -C_\delta u$

$$f(\cdot, u) \in L^\infty(\Omega), \qquad f(x,u) \leq C(u+1)$$

$$a_0(x) = \limsup_{u \to 0^+} \frac{f(x,u)}{u} \qquad \textit{and} \qquad a_\infty(x) = \liminf_{u \to +\infty} \frac{f(x,u)}{u}.$$

Assume

$$\lambda_1(-\Delta - a_0) < 0 < \lambda_1(-\Delta - a_\infty).$$

Then there exist **at least** *a positive solution.*

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$$V(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)$$

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The almost periodic case

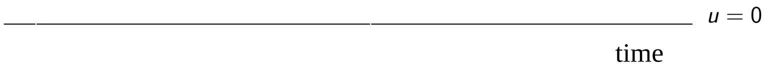
- If

$$\frac{f(t,x,u)}{u} \quad \text{is } \textbf{nonincreasing} \text{ in } u \geq 0.$$

then S has a **unique** positive globally attractive fixed point.

[Shen, Yi, J.Math.Biol.(1998)]

The special positive solution



The trivial sate and the special solution

At each time the state $u_S(t)$ is the important one because it is the pullback attractor

But also u_S attracts forwards

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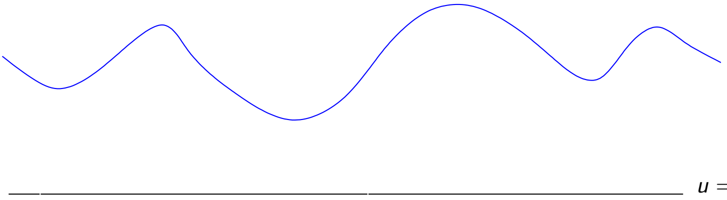

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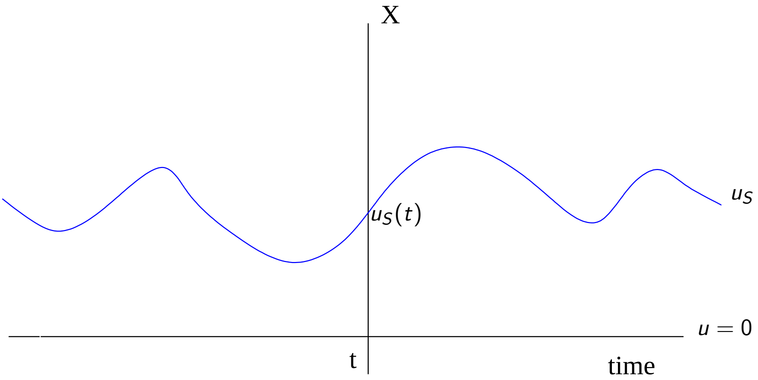

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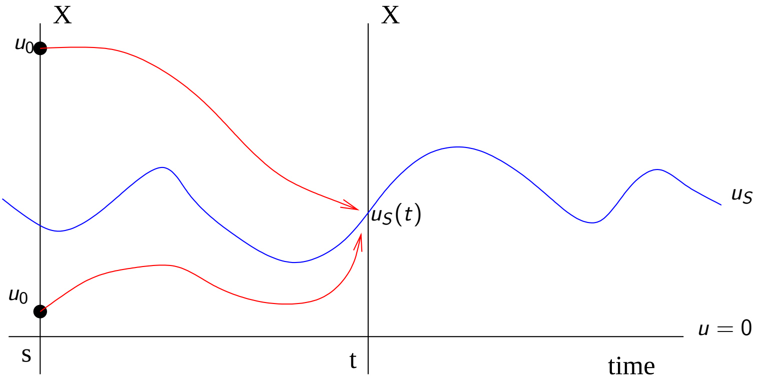

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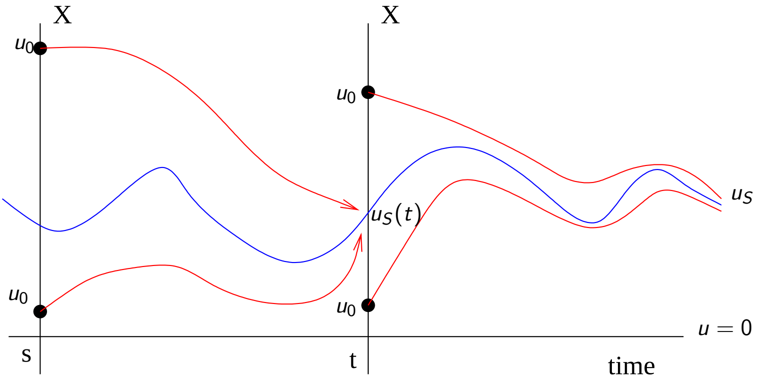

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A result with restrictions

[Langa, Robinson, Suarez, Int.J.Bif.Chaos (2005)]

$$f(t,x,u) = \lambda u - n(x,t)u^{\rho}$$

$$\lambda > \lambda_1^D(\Omega)$$

$$0 < a_0 \leq n(x,t) \leq A_0, \qquad \text{with} \qquad A_0 \leq \rho a_0.$$

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Hence there are two complete positive trajectories

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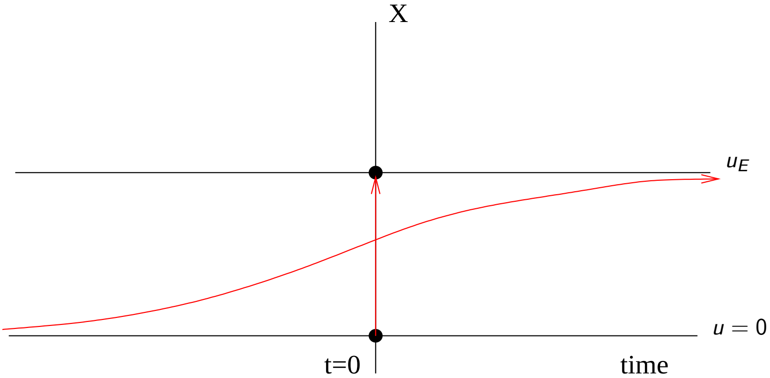

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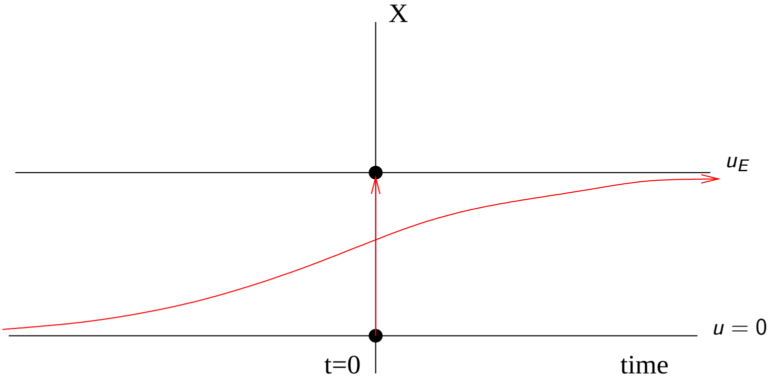

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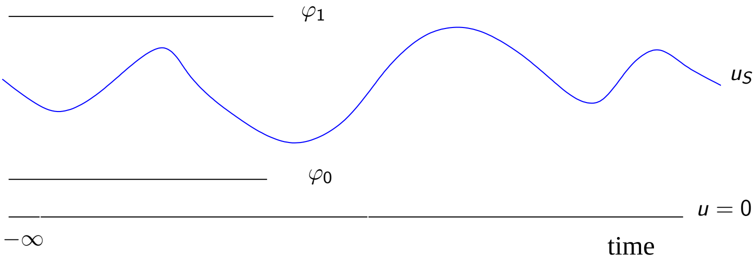
Uniqueness of CBNDS: complete bounded nondegenerate solution



$$\varphi_0, \varphi_1 \in C_0^1(\overline{\Omega}), \qquad 0 < \varphi_0 \le u_S(t) \le \varphi_1, \quad t << -1$$

Theorem (ARB-A.Vidal–López, 2005)

Assume

$$f(t,x,u) = m(x,t)u - n(x,t)u^{\rho}, \quad n(x,t) \geq 0$$

$$n(x,t) \geq a_0 > 0 \quad in \quad \overline{\Omega} \times \mathbb{R} \quad (can\ be\ weakened)$$

$$m \in L^{\infty}(\mathbb{R}, L^{p}(\Omega)) \quad with \quad p > N$$

for $t << -1$

$$m(x,t) \geq M(x) \qquad n(x,t) \leq N(x)$$

such that

$$f_0(x) = M(x)u - N(x)u^{\rho} \leq f(t,x,u)$$

is of "Brezis-Oswald" type. Then there exists a unique CBPND trajectory

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Definition

i) If X a Banach space, the family $T(t,s) \in \mathcal{L}(X)$, is an evolution operator
a) $T(t,t) = I$ for all $t \in \mathbb{R}$,
b) $T(t,s)T(s,r)u = T(t,r)u$ for all $r \leq s \leq t$, $u \in X$, and
c) $u \mapsto T(t,r)u$ is continuous in X for $t > r$.

ii) $T(t,s)$ is **exponentially stable** *of exponent $\beta > 0$ if for some $M > 0$*

$$||T(t,s)||_{\mathcal{L}(X)} \leq M\mathrm{e}^{-\beta(t-s)} \quad \text{for all } t > s.$$

Return