Image Restoration and Mathematical Aspects of Total Variation Based $u + v$ decompositions

V. Caselles

[1] Universitat Pompeu Fabra, Barcelona

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Introduction: Notation

- $u : [0,1] \times [0,1] \to I\!R$ is the undistorted image
- f is the observed image
- K is a convolution operator with kernel k. The kernel k takes into account several effects: the optical system, the sensors, motion blur, etc.
- n is an additive (Gaussian) white noise with zero mean and variance σ^2.

Image Acquisition Model: The equation relating u to f can be written as

$$f = Ku + n \qquad \text{(IM)}$$

Denoising Problem

If $K = I$, the (IM) is

$$f = u + n \qquad \text{(IM)}_{K=I}.$$

Denoising Problem: We know f and the statistics of the noise. We want to recover u.

Restoration Problem

Problem: Recover u from f and the knowledge about the kernel K and the statistics of the noise.

This is an ill-posed problem

- K need not be invertible
- Even if K^{-1} exists, applying it to both sides of (IM) we obtain

$$K^{-1}f = u + K^{-1}n$$

Writing $K^{-1}n$ in the Fourier domain, we have

$$K^{-1}n = \left(\frac{\hat{n}}{\hat{k}}\right)^{\vee}.$$

We see that the noise may blow up at the frequencies for which $\hat{k}$ vanishes or it becomes small.

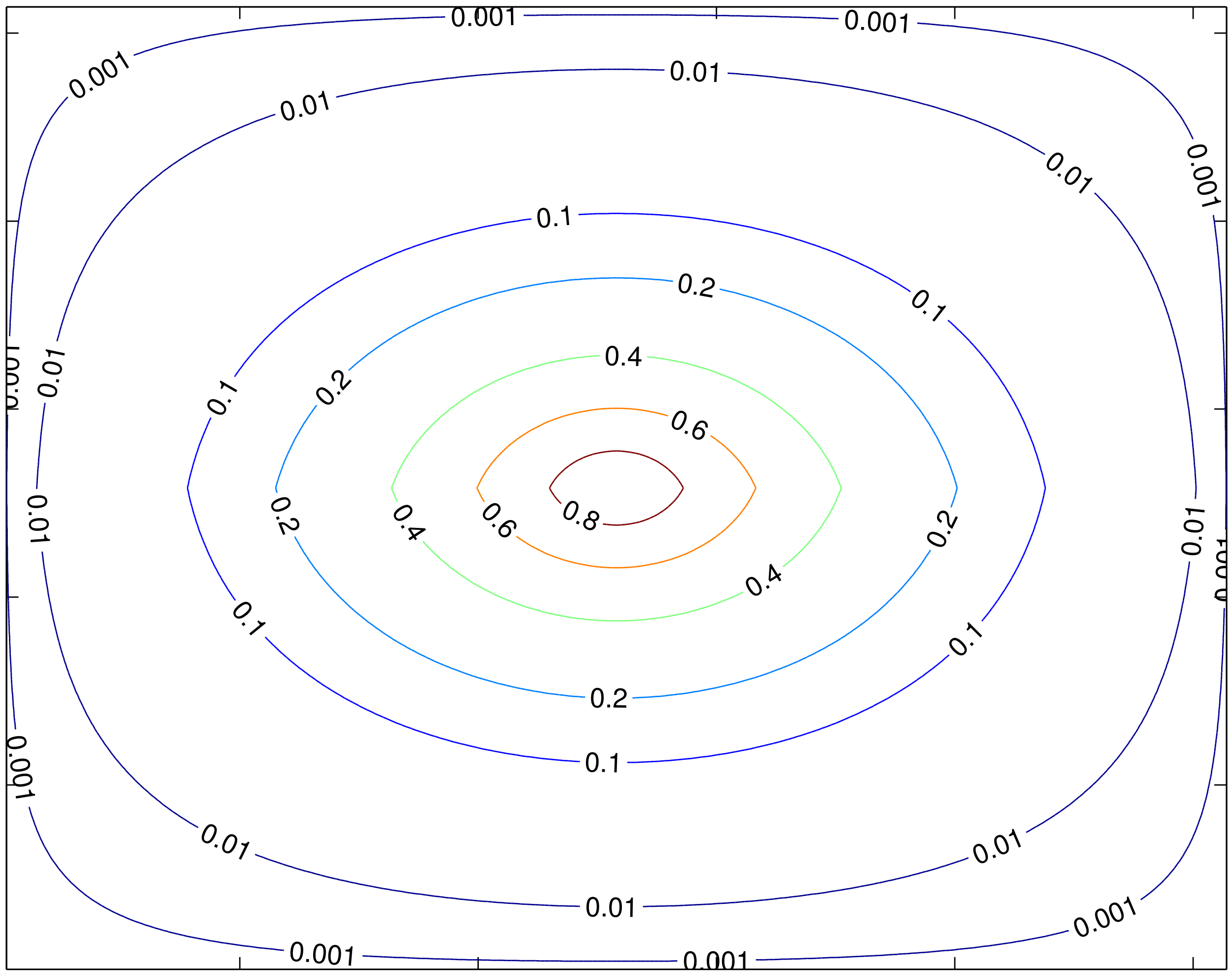

Figure 1: Some level lines in $[-1/2, 1/2]^2$ of MTF (the FT of K)

Regularization methods

First: interpret (IM) in an integral sense

$$\int_{[0,1]^2} (Ku - f)^2 \, dx \leq \sigma^2 \qquad \text{(IM*)}$$

• H^1 regularization: choose between the possible solutions of (IM*) the one which minimizes

$$\int_{[0,1]^2} |\nabla u|^2 \, dx$$

Not satisfactory: unable to resolve discontinuities (edges) and oscillatory patterns. Functions in $W^{1,2}$ are not good models for images.

• Total Variation Regularization: from all functions satisfying (IM*) choose the one which minimizes

$$\int_{[0,1]^2} |Du|$$

Model proposed by Osher-Rudin-Fatemi (1992). The underlying functional model for images are BV functions: contain geometric image sketches.

From constrained to Unconstrained Formulation

Typical approach: solve the unconstrained minimization problem

$$\text{Minimize} \int_{[0,1]^2} |\nabla u| + \frac{\lambda}{2} \int_{[0,1]^2} (Ku - f)^2, dx$$

The constraint has been introduced with a Lagrange multiplier. Formally. the Euler-Lagrange equations are :

$$-\text{div}\left(\frac{Du}{|Du|}\right) + \lambda K^*(Ku - f) = 0$$

with Neumann boundary conditions.

Restoration Experiments

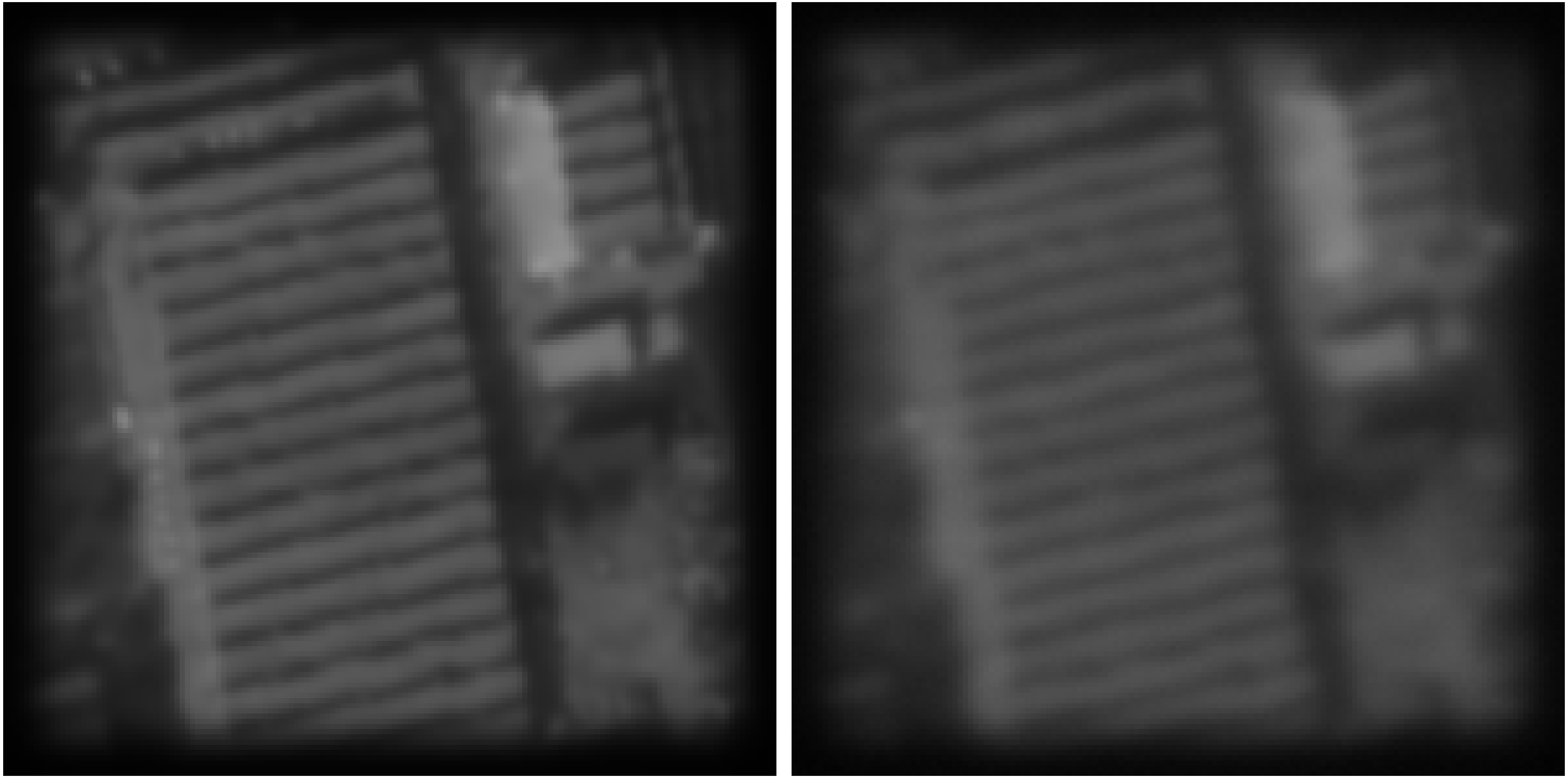

Figure 2: Left: Reference image. Right: Data

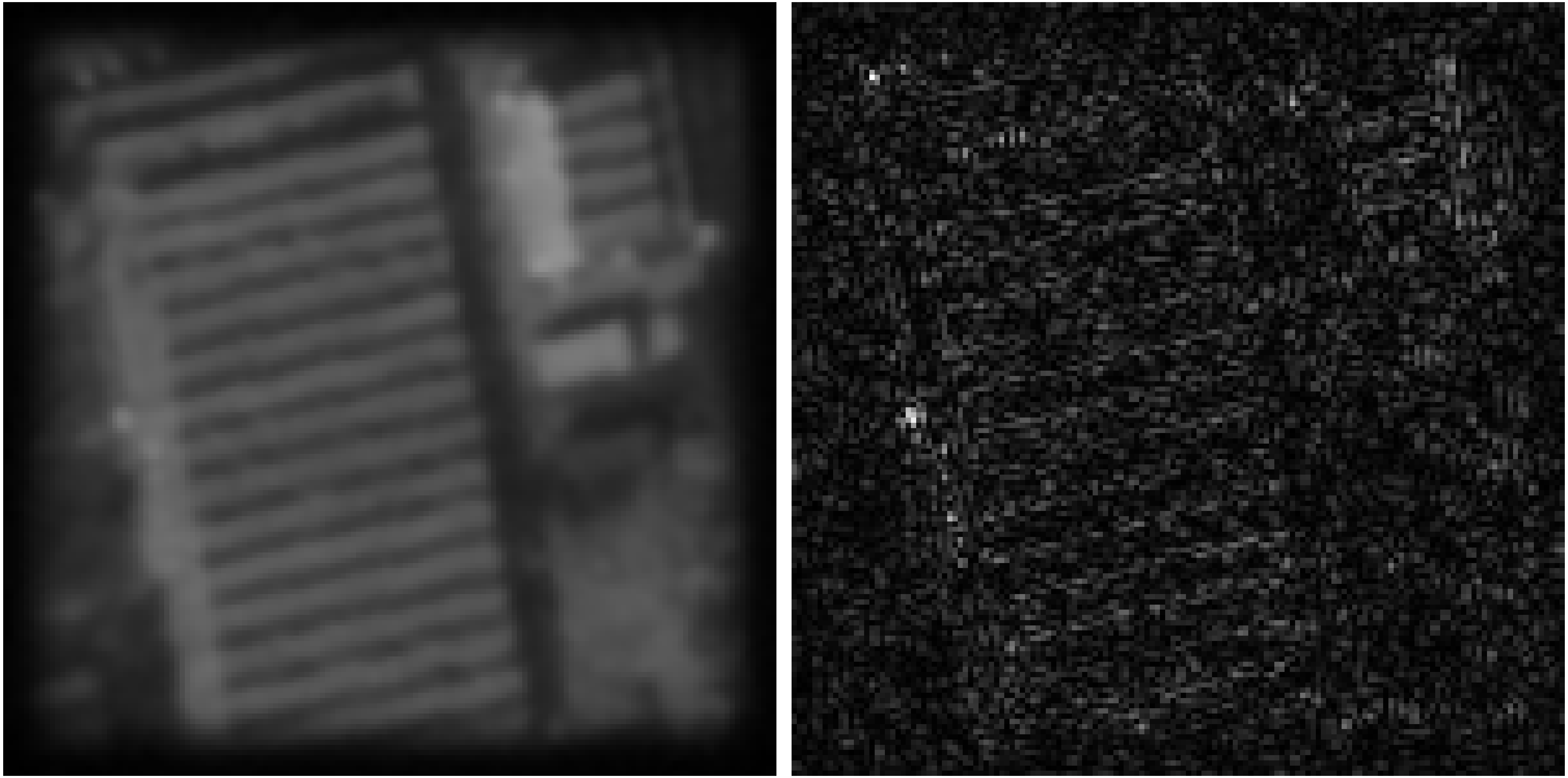

Figure 3: Left: Restored image. Right: Error

Figure 4: Left: Reference. Rigt: Data

Figure 5: Restored image

Denoising, $u+v$ decompositions

• If $K = I$: TV denoising model

$$\text{Minimize} \int_{[0,1]^2} |\nabla u| + \frac{\lambda}{2} \int_{[0,1]^2} (u-f)^2 dx \qquad \text{(DP)}$$

The Euler Lagrange equations are:

$$f = u - \frac{1}{\lambda} \text{div} \left(\frac{Du}{|Du|} \right) \quad + \text{NBC}$$

Not a good denoising model: residuals have structure.

• Second Life after a different interpretation given by Y. Meyer:

Write $v = -\frac{1}{\lambda} \text{div} \left(\frac{Du}{|Du|} \right)$ and write $f = u + v$ where u and v solve:

$$\text{Minimize} \int_{[0,1]^2} |\nabla u| + \frac{\lambda}{2} \|v\|_2^2 \qquad \text{whith } f = u + v.$$

u is the geometric sketch of the image

v is the textured part

$u + v$ decompositions

Write

$$v = -\frac{1}{\lambda}\operatorname{div} z \quad \text{with } z \cdot Du = |Du|,\ |z|_\infty \leq 1$$

The $u + v$ decomposition is obtained by minimizing:

$$\text{Minimize} \int_{[0,1]^2} |\nabla u| + \frac{\lambda}{2}\|v\|_2^2 \qquad \text{whith } f = u + v.$$

Criticism: Not a good model. Reasons: experimental and theoretical.

Our purpose: Give examples which show its defects.

- New Meyer's proposal; Better Model :

$$\text{Minimize} \int_{[0,1]^2} |\nabla u| + \lambda\|v\|_* \qquad \text{whith } f = u + v.$$

Notations: $\|\|_*$ denotes de norm in BV^* (the dual of BV).

Experiments (images courtesy of A. Chambolle)

Figure 7: Left: Noisy image. Rigt: Result

Figure 8: Denoising for different values of λ

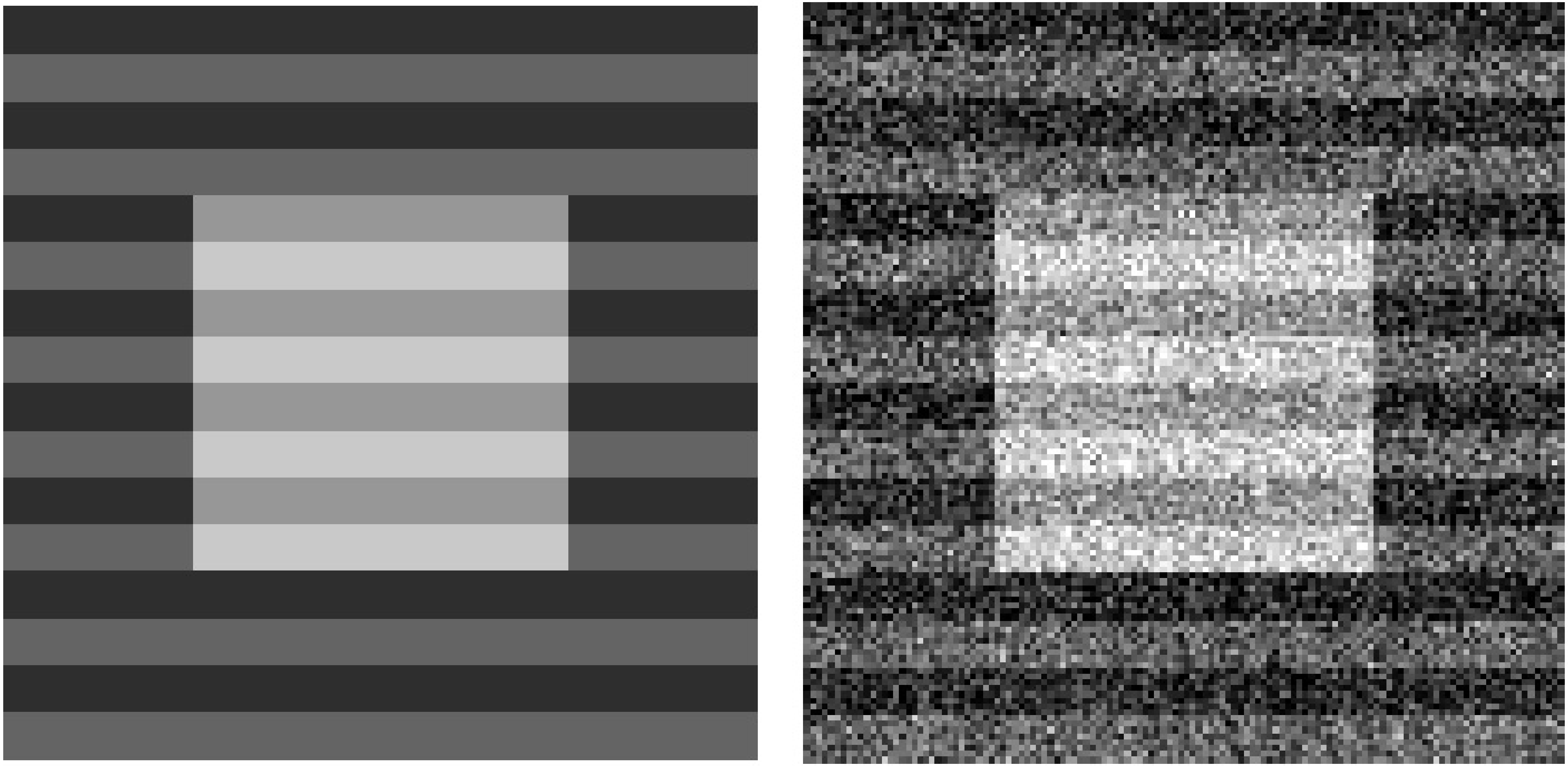

Figure 9: Left: Reference. Rigt: Noisy Data

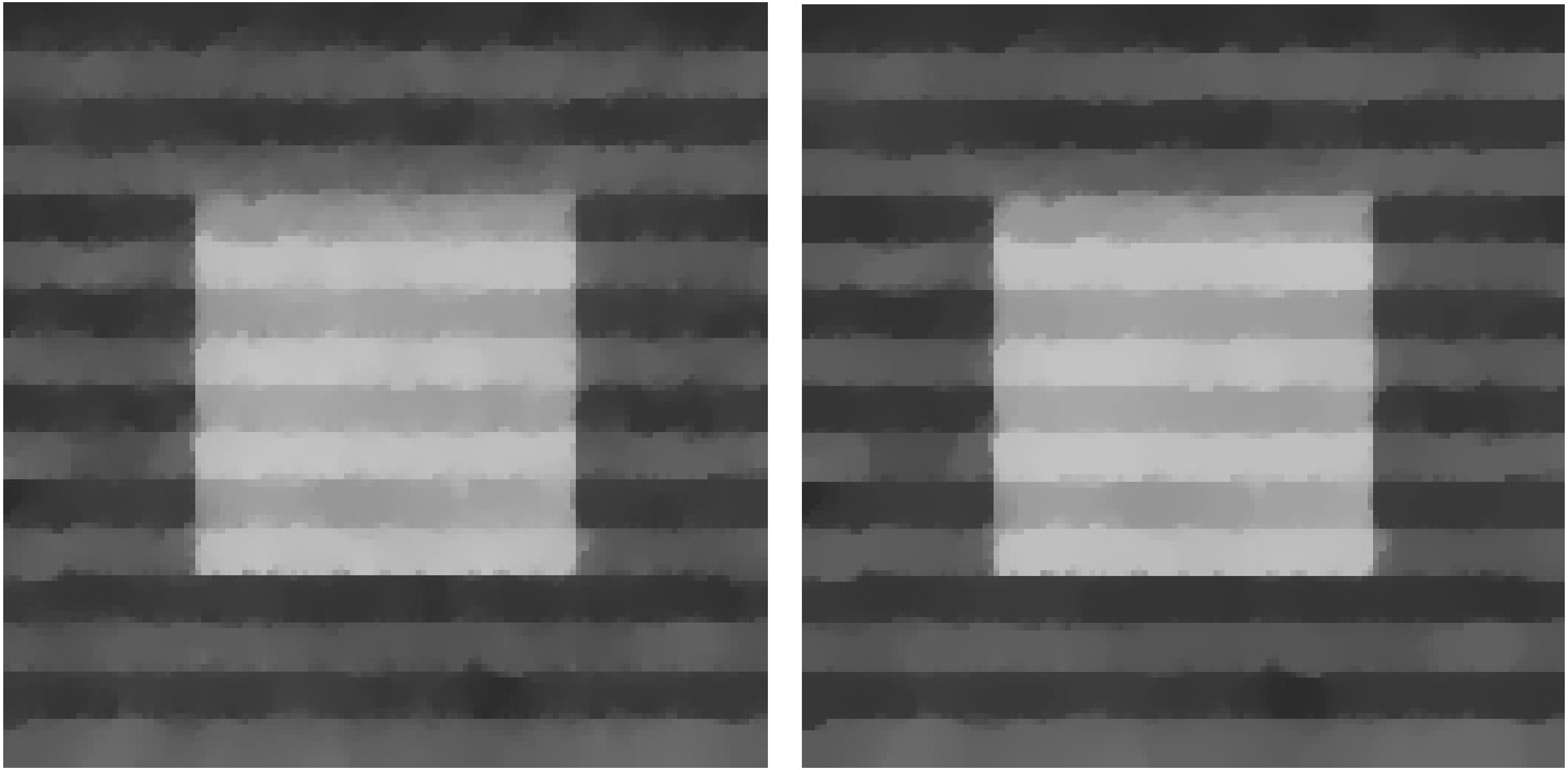

Figure 10: Left: Result ROF model. Rigt: Result Meyer's model

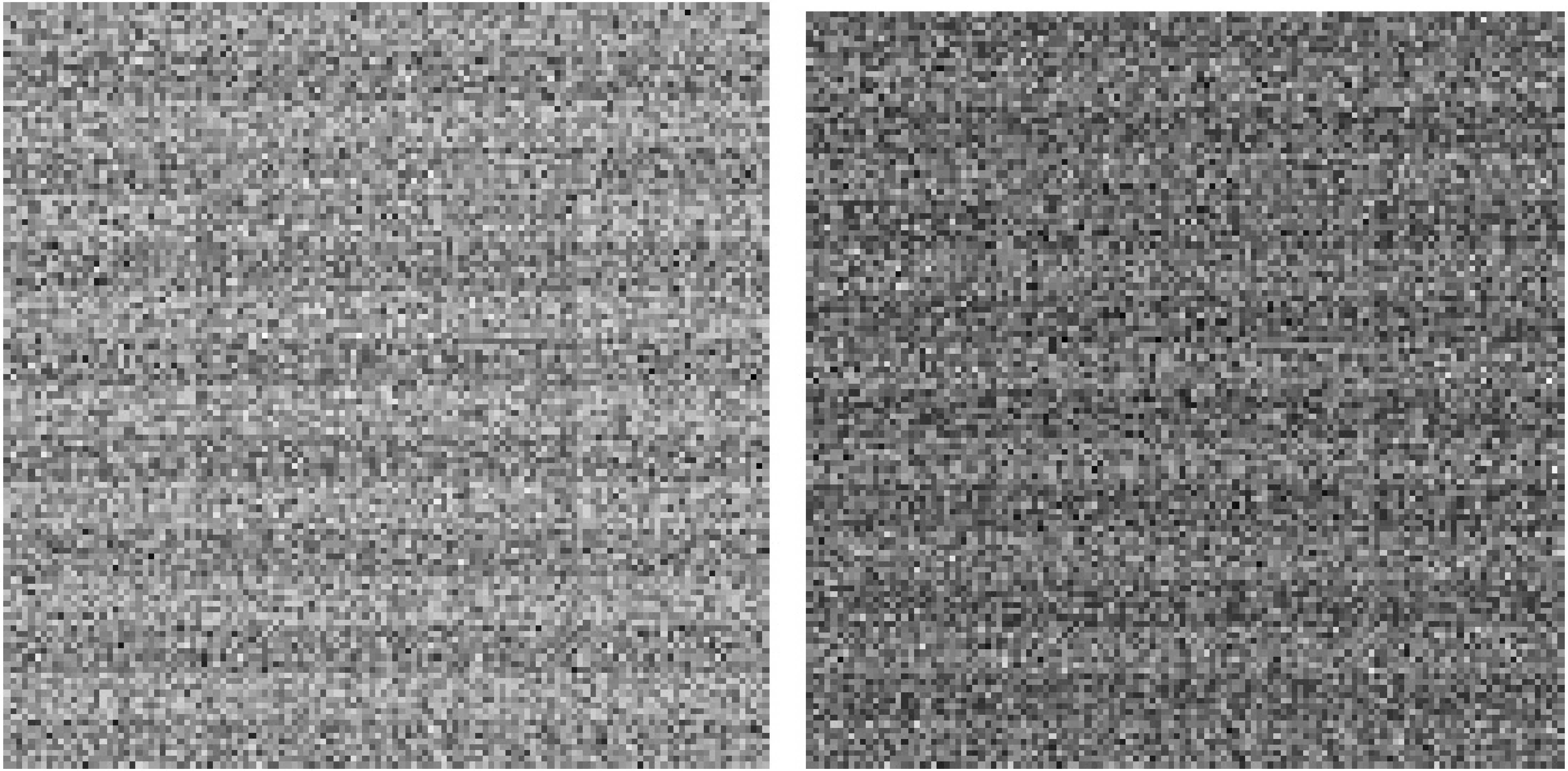

Figure 11: Differences Original with Result

Our purpose

• Find data f for which we can compute explicitly its $u+v$ decomposition to show some defects of the model.

Some References

• The study of the operator $-\mathrm{div}\left(\frac{Du}{|Du|}\right)$ has been done by F. Andreu, V.C., and J.M. Mazón in several contexts (Int. Diff. Eq., JFA, Birkhauser ...)

• Explicit $u+v$ decompositions in $I\!R^2$: G. Bellettini, V.C., M. Novaga (JDE,SIAM J MA)

• More explicit solutions in $I\!R^2$ and in $I\!R^N$: F. Alter, V.C., A. Chambolle (IFB, Math. Ann)

A basic result to construct explicit solutions

Proposition (BCN, 2002) Let $\overline{u}$ be a solution of the Eigenvalue problem

$$-\mathrm{div}\left(\frac{Dw}{|Dw|}\right) = w \qquad \text{(EP)}.$$

Let $\lambda > 0$, $b \in \mathbb{R}$, and $a := \mathrm{sign}(b)(|b| - \lambda^{-1})^{+}$.

Let $f = b\overline{u}$. The solution of (DP) is $u = a\overline{u}$.

Sketch of proof : Assume that $0 < \lambda^{-1} \leq b$, then $a = b - \lambda^{-1}$. Then

$$f = b\overline{u} = a\overline{u} + \lambda^{-1}\overline{u} = a\overline{u} - \lambda^{-1}\mathrm{div}\left(\frac{D\overline{u}}{|D\overline{u}|}\right) = u - \lambda^{-1}\mathrm{div}\left(\frac{Du}{|Du|}\right).$$

A variant of the basic result

Proposition Let $u_i \in BV(I\!R^N)$, $u_i \geq 0$, be functions with disjoint support.

Assume that u_i, $\sum_{i=1}^m u_i$ are solutions of (EP).

Let $f = \sum_{i=1}^m b_i u_i$ ($b_i \in I\!R$, $\lambda > 0$).

Then $u = \sum_{i=1}^m \mathrm{sign}(b_i)(|b_i| - \lambda^{-1})^+ u_i$ is the solution of (DP).

Solutions of the Eigenvalue Problem

Theorem (BCN, JDE 2002) Let $C \subseteq \mathbb{R}^2$ be a set of finite perimeter. Assume that C is connected. Then

(i) $v = \lambda \chi_C$ is a solution of the (EP)

(ii) $\lambda = \lambda_C := \frac{P(C)}{|C|}$, C is convex, $C^{1,1}$ and

$$C = \operatorname{argmin} P(X) - \lambda_C |X| \qquad X \subseteq C$$

(iii) $\lambda = \lambda_C$, C is convex, $C^{1,1}$ and

$$\operatorname{esssup}_{x \in \partial C} \kappa_{\partial C}(x) \leq \lambda_C$$

Solutions of the Eigenvalue Problem

Proof: $(i) \Rightarrow (ii)$ If $\lambda\chi_C$ is a solution of (EP), there exists a vector field z with $|z| \leq 1$ such that

$$-\mathrm{div}\, z = \lambda\chi_C \qquad \text{and } z \cdot D\chi_C = |D\chi_C|.$$

Multiplying this PDE by χ_C and integrating by parts, we have

$$\lambda|C| = \int |D\chi_C| = P(C) \longrightarrow \lambda = \frac{P(C)}{|C|}.$$

Now, we multiply by χ_D, $D \subseteq I\!R^2$ a set of finite perimeter. We obtain:

$$\lambda_C|C \cap D| = \int z \cdot D\chi_D \leq P(D) \longrightarrow \lambda_C \leq \frac{P(D)}{|C \cap D|}.$$

Proof continued

We have

$$\lambda_C \leq \frac{P(D)}{|C \cap D|} \qquad \forall D$$

Taking $D \subseteq C$ we have

$$\lambda_C \leq \frac{P(D)}{|D|} \quad \forall D \subseteq C \longrightarrow C = \mathrm{argmin} P(X) - \lambda_C |X|$$

Taking $D \supseteq C$ we obtain

$$P(C) \leq P(D) \quad \forall D \supseteq C \longrightarrow P(C) \leq P(co(C)) \longrightarrow C \text{ is convex.}$$

$(ii) \Rightarrow (iii)$ If $C = \mathrm{argmin} P(X) - \lambda_C |X|$, then $\kappa_{\partial C} \leq \lambda_C$ on ∂C.

$(iii) \Rightarrow (ii)$ Based on a result of Giusti ($N = 2$). Other proofs by BCN and Kawohl-Lachand Robert.

$(ii) \Rightarrow (i)$ Use the coarea formula.

Solutions of (EP): non-connected sets

Theorem (BCN, JDE 2002) Let $\Omega \subseteq I\!R^2$ be a set of finite perimeter. Let $\Omega = C_1 \cup \ldots \cup C_m$, $b_i > 0$. Then

$u = \sum_{i=1}^{m} b_i \chi_{C_i}$ is a solution of the (EP) if and only if

- $b_i = \frac{P(C_i)}{|C_i|}$ $\forall i$
- C_i are convex sets, $C^{1,1}$

$$\operatorname{esssup}_{x \in \partial C} \kappa_{\partial C_i}(x) \leq \frac{P(C_i)}{|C_i|} \quad \forall i$$

- If $E = \operatorname{argmin} P(X)$ for $X \supseteq \cup_i C_i$, then $P(E) = \sum_{i=1}^{m} P(C_i)$.

Solutions of (EP): non-connected sets

• If $P(co(C_1 \cup C_2)) < P(C_1) + P(C_2)$, there is interaction of the two sets.

• If $P(co(C_1 \cup C_2)) \geq P(C_1) + P(C_2)$, both sets evolve independently without interaction.

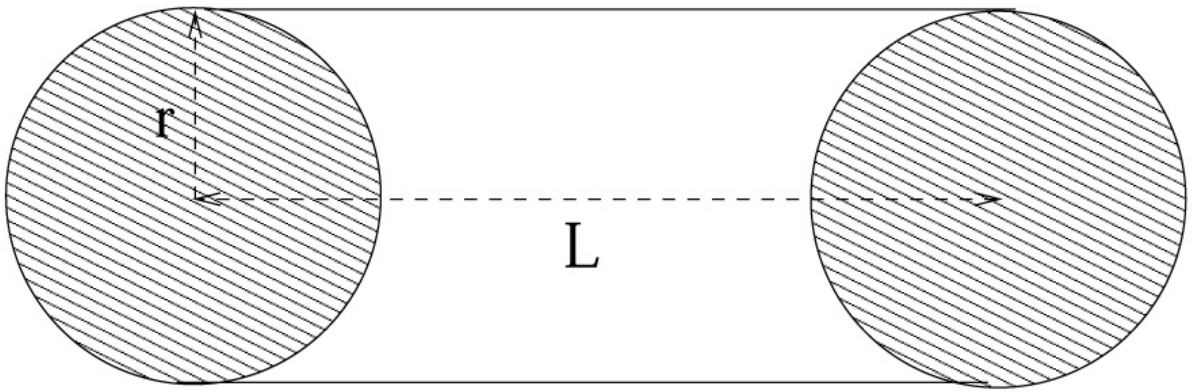

Figure 12: Two balls

Solutions of (EP) which are Towers of convex sets

Theorem (BCN, 2002) Let $K_0, K_1 \subseteq \mathbb{R}^2$ be two convex sets with $\overline{K_1} \subseteq K_0$. Let

$$J := \frac{P(K_0) - P(K_1)}{|K_0 \setminus K_1|}.$$

If

$$\mathrm{esssup}_{x \in \partial K_0} \kappa_{\partial K_0}(x) \leq J$$

$$\mathrm{essinf}_{x \in \partial K_1} \kappa_{\partial K_1}(x) \geq J, \qquad \mathrm{esssup}_{x \in \partial K_1} \kappa_{\partial K_1}(x) \leq \lambda_{K_1}$$

then $v = \lambda_{K_1} \chi_{K_1} + J \chi_{K_0 \setminus K_1}$ is a solution of (EP).

Solutions of (EP) in $W^{1,1}_{loc}(I\!R^N) \cap L^\infty(I\!R^N)$

Theorem (BCN, 2002) Classification of solutions u of (EP) such that $T_k(u) \in W^{1,1}_{loc}(I\!R^N)$ $\forall k > 0$, and are in $L^\infty_{loc}(I\!R^N)$ near any level set.

Solutions are such that: the connected components of $[u > t]$ are circles of radius $\frac{1}{t}$.

Solutions of (EP) in $I\!R^N$

Theorem (ACCh, 2003) Let $C \subseteq I\!R^N$ be a convex set of class $C^{1,1}$. The following conditions are equivalent:

- $u = \lambda_C \chi_C$ is a solution of (EP).
- $C = \operatorname{argmin}_{X \subseteq C} P(X) - \lambda_C |X|$
- $(N-1)\mathbf{H}_C \leq \lambda_C$.

A Technical Tool

An important tool to prove last Theorem is :

Proposition (ACCh, 2003) Let $C \subseteq \mathbb{R}^N$ be a convex set. If $u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a solution of

$$\text{Minimize} \int_{\mathbb{R}^N} |\nabla u| + \frac{\lambda}{2} \int_{\mathbb{R}^N} (u - \chi_C)^2 dx \qquad \text{(DPC)}$$

then the level set $[u \geq s]$, $0 < s \leq 1$, is a solution of

$$\text{Min}_{X \subseteq C} P(X) - \lambda(1-s)|X|$$

Solutions of (EP) in $\mathbb{R}^N$

Some explanation: If $(N-1)\mathbf{H}_C \leq \lambda_C$ and C is not a minimizer of

$$\mathrm{argmin}_{X\subseteq C} P(X) - \lambda_C |X|$$

then it would be a minimizer of

$$\mathrm{argmin}_{X\subseteq C} P(X) - \mu |X| \qquad (P)_\mu$$

for some $\mu > \lambda_C$ and would be approximated by minimizers C_n of $(P)_{\mu_n}$ with $\mu_n \uparrow \mu$. Since

• The sets C_n are convex (consequence of previous result and Korevaar's concavity results)

• Since $C_n \neq C$, ∂C_n has points with $(N-1)\mathbf{H}_{C_n} = \mu_n > \lambda_C$.

$\Longrightarrow$ ∂C contains points where $(N-1)\mathbf{H}_C > \lambda_C$, a contradiction.

Connections with an Isoperimetric problem

Thus, level sets of (DPC) are related to the solutions of

$$\mathrm{Min}_{X \subseteq C} P(X) - \mu|X| \qquad \mu > 0.$$

which, in turn, is related to

$$\mathrm{Min}_{X \subseteq C, |X| = V} P(X) \qquad (IsoPV)$$

where V is a fixed volume.

Theorem (ACCh, 2003) Let $C \subseteq \mathbb{R}^N$ be a convex set of class $C^{1,1}$. Then for any value of $V \in [|K|, |C|]$ there is a unique convex solution of the isoperimetric problem (IsoPV).

K is the "maximal Cheeger" set ("maximal calibrable" set) contained in C.

General convex sets

(ACCh, IFB 2005) Assume that C is a bounded convex set in $\mathbb{R}^2$:

• There is a calibrable set $C_R \subseteq C$ such that $\partial C \setminus \partial C_R$ is formed by arcs of circle of radius R such that $\frac{1}{R} = \frac{P(C_R)}{|C_R|}$

• For each $x \in C \setminus C_R$ it passes an arc of circle of radius $r(x)$ and those circles fiber $C \setminus C_R$.

Then $u(x) = (1 - \frac{\lambda^{-1}}{r(x)})^+ \chi_C$ is the solution of (DPC)

$$\text{Minimize} \int_{\mathbb{R}^N} |\nabla u| + \frac{\lambda}{2} \int_{\mathbb{R}^N} (u - \chi_C)^2 dx \qquad \text{(DPC)}$$

This result can be extended to $\mathbb{R}^N$ with suitable adaptations.

Open Questions

- Restoration of textures: modeling based on functional norms ?
- Restoration when the kernel vanishes at the interior of the frequency domain.
- Solutions of the eigenvalue problem (in $\mathbb{R}^N$):

$$-\mathrm{div}\left(\frac{Dw}{|Dw|}\right) = w \qquad \text{(EP)}.$$

- Is the solution of the isoperimetric problem

$$\mathrm{Min}_{X \subseteq C, |X| = V} P(X) \qquad (IsoPV)$$

a convex set (C being convex) ?

- Assume that C is convex, is the Cheeger set contained in C unique ?