PDEs with highly variable coefficients: modeling, asymptotics & computation

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Collaborators

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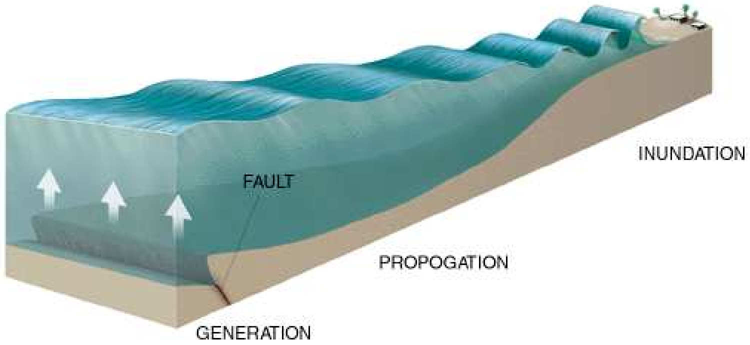
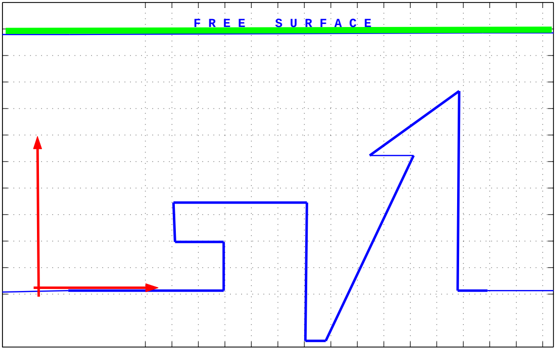
PHYSICAL MODELING

PHYSICAL MODEL: Long propagation distances + detailed TOPOGRAPHY

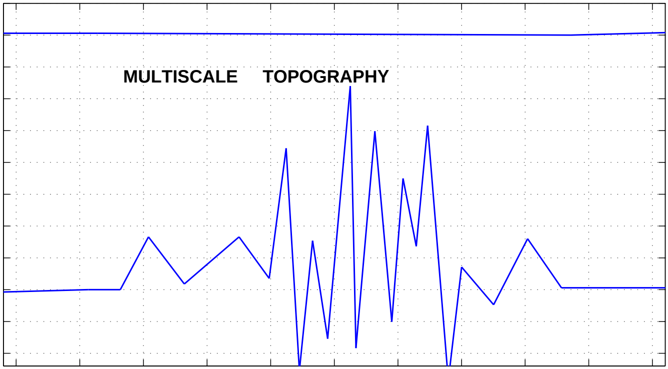


Scientific American '99

We might consider SUBMERGED STRUCTURES



A typical geometry for MULTISCALE TOPOGRAPHIES



- FLUID: inviscid; FLOW: incompressible, irrotational.
- MATH/Scaling: Large time, pulse propagation over a rapidly varying medium (3 scales).
- REGIME: weakly NONLINEAR, weakly DISPERSIVE $\Rightarrow$ Solitary waves in disordered media
- 2D EULER eqns. $\Rightarrow$ POTENTIAL THEORY $((u, v) \equiv \nabla\phi)$

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INTRODUCTION: NonLinear Potential Theory

The dimensionless equations are

$$\beta\phi_{xx} + \phi_{yy} = 0, \qquad \text{in} \qquad \Omega \equiv \text{FLUID BODY},$$

with nonlinear boundary conditions at the ...FREE SURFACE

$$\begin{cases} \phi_t + \frac{\alpha}{2}(\phi_x^2 + \frac{1}{\beta}\phi_y^2) + \eta & = 0 \\ \eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y & = 0 \end{cases} \qquad \text{in} \qquad y = \alpha\,\eta(x,t)$$

and a Neumann condition

$$\frac{\beta}{\gamma}h'(\tfrac{x}{\gamma})\phi_x + \phi_y = 0 \qquad \text{along} \qquad y = -\sqrt{\beta}h(\tfrac{x}{\gamma}),$$

namely ....the highly variable TOPOGRAPHY h.

$\alpha \equiv (amplitude/depth)$, $\beta \equiv (depth/wavelength)^2$, $\gamma \equiv (disorder/wavelength)$

NONLINEARITY DISPERSION MULTISCALE

MATH MODELING/PDE asymptotics

GOAL: reduce the system of PDEs (2D→1D) in particular when

$\alpha \ll 1 \quad \Leftrightarrow \quad$ weakly nonlinear free surface waves

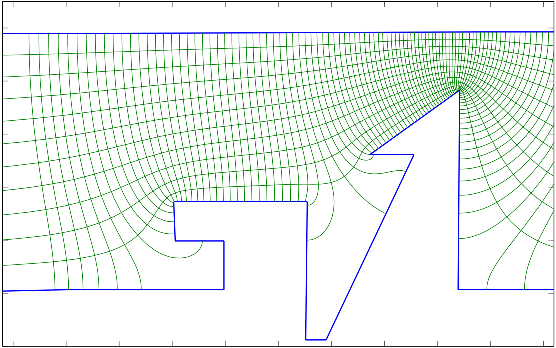
$\gamma \ll 1 \quad \Leftrightarrow \quad$ rapidly varying topographies

(A) $\beta \ll 1$ **(B)** any β

Key idea: use curvilinear coordinate system

$(\xi, \zeta) \Leftarrow$ Conformal Mapping

Toby Driscoll's **Schwarz-Christoffel Toolbox/MATLAB** $\Rightarrow$ **FREE!!**



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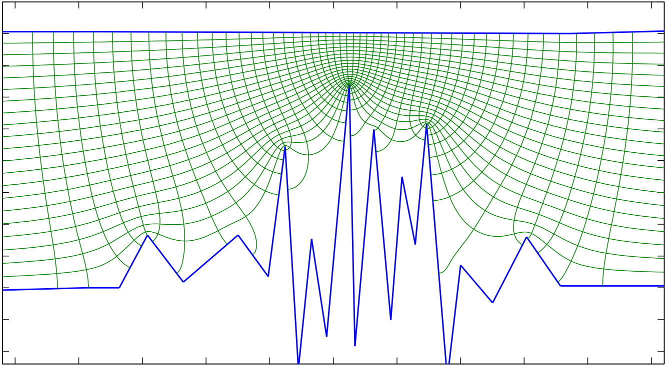


CURVILINEAR COORDINATES:

N. SIAP '03

$$\phi_{\xi\xi} + \phi_{\zeta\zeta} = 0, \qquad -\sqrt{\beta} < \zeta < S(\xi, t).$$

The free surface (FS) conditions are $\eta(x,t) \approx N(\xi(x,0),t)/M(\xi)$

$$N_t + \frac{\alpha}{|J|}\phi_\xi N_\xi - \frac{1}{|J|\sqrt{\beta}}\phi_\zeta = 0.$$

$$\phi_t + \frac{\alpha}{2|J|}(\phi_\xi^2 + \phi_\zeta^2) + \eta = 0.$$

Note that $\phi_\zeta = 0$ at $\zeta = -\sqrt{\beta}$.

$$(\partial_{\xi\xi} + \partial_{\zeta\zeta}) = |J|^2 \Delta_{xy} \Rightarrow |J| \equiv (y_\xi^2 + y_\zeta^2)_{|FS} \approx \quad y_\zeta^2(\xi, 0) + O(\varepsilon^2) \text{ (WEAKLY NONLIN.)}$$

FS metric coefficient is $M(\xi; \sqrt{\beta}, \gamma) \equiv y_\zeta(\xi, 0)$, where

$$M(\xi; \sqrt{\beta}, \gamma) = \frac{\pi}{4\sqrt{\beta}} \int_{-\infty}^{\infty} \frac{h(x(\xi_o, -\sqrt{\beta})/\gamma)}{\cosh^2 \frac{\pi}{2\sqrt{\beta}}(\xi_o - \xi)} d\xi_o.$$

Power series near the bottom (shifted to be at) $\zeta = 0$ Whitham 1974

$$\phi(\xi,\zeta,t) = \sum_{n=0}^{\infty} \zeta^n \; f_n(\xi,t).$$

The velocity potential (satisfies LAPLACE + NEUMANN cond.)

$$\phi(\xi,\zeta,t) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(2n)!} \zeta^{2n} \frac{\partial^{2n} f(\xi,t)}{\partial \xi^{2n}} \approx \sum_{n=0}^{N} [\ldots]$$

We have that

$$C^2(k) = \frac{\omega^2}{k^2} = \frac{1}{\sqrt{\beta}k} tanh(\sqrt{\beta}k)$$

(PHASE SPEED)^2 $$\approx 1 - \frac{1}{3}(\sqrt{\beta}k)^2 + \tfrac{2}{15}(\sqrt{\beta}k)^4 - \tfrac{17}{315}(\sqrt{\beta}k)^6 + O((\sqrt{\beta}k)^8)$$

Truncate the dispersion relation through Padé Approximation.

Differentiating ϕ with respect to ξ and evaluating the velocity at an INTERMEDIATE depth $\zeta = Z_0 \in [0,1]$

$$\phi_\xi(\xi, Z_0, t) \equiv u(\xi, t) = f_\xi - \frac{\beta}{2} {Z_0}^2 f_{\xi\xi\xi} + O(\beta^2)$$

FREE SURFACE CONDITIONS reduce to the BOUSSINESQ-family of equations

$$M(\xi)\eta_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi + \frac{\beta}{2}\left[\left(\mathbf{Z_0}^2 - \frac{1}{3}\right) u_{\xi\xi}\right]_\xi = 0$$

$$u_t + \eta_\xi + \alpha \left(\frac{u^2}{2M^2(\xi)}\right)_\xi + \frac{\beta}{2}(\mathbf{Z_0}^2 - 1) u_{\xi\xi t} = 0$$

N. SIAP '03: depth-averaged Boussinesq system $\Leftrightarrow Z_0 = \sqrt{1/3}$.

Muñoz & N., SIAP '04: Apparent diffusion due to random topography

Muñoz & N., SIAM MMS '05: Stiff microscale forcing $\Leftarrow$ Conf.Coord. preconditions Peregrine '67

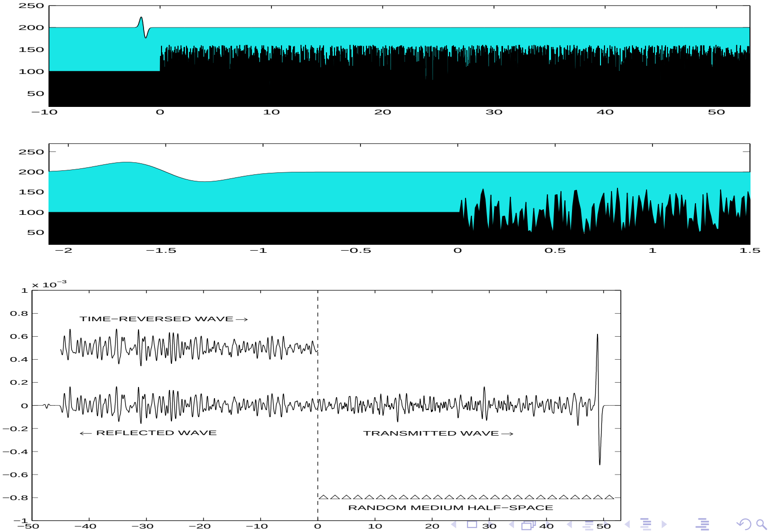
Fouque, Garnier & N., SIAP '04: Time-reversal and refocusing $\Rightarrow$ waveform inversion

Fouque, Garnier, Muñoz & N., PhysRevLett '04: Time-reversing solitary waves

Overview of SOLUTION asymptotics

SETUP for THEORY and SIMULATIONS:

Typical wave profiles: Gaussian, dGaussian/dx and Solitary wave.



RESULTS for...

Depth-averaged velocity $\Leftrightarrow Z_0 = \sqrt{1/3}$

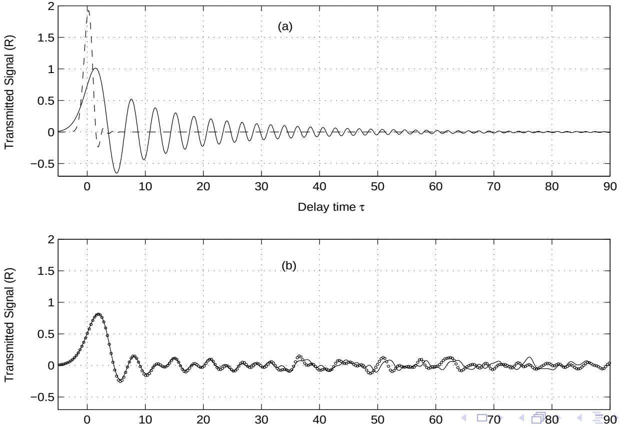
Linear Dispersive Gaussian pulse: APPARENT DIFFUSION,

Muñoz & N., SIAP '04 ⇒ deterministic **Fourier integral** capturing **coherent front** and **disordered coda**.

DISORDER ⇒ **effective decrease in Dispersion:** Fouque, Garnier & N., SIAP '04; Garnier & N. submitted '05.

LINEAR NONLINEAR

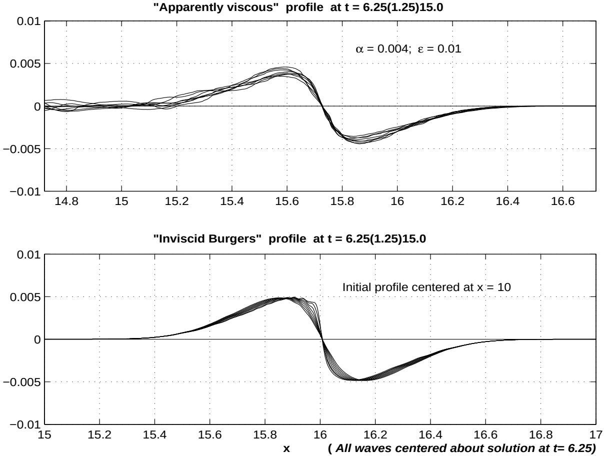
Wave elevation $\equiv \eta(x,t) \approx \eta_0 * K \sim \eta_0 * G,$ $G \equiv$ Gaussian kernel



NonLinear Shallow Water system w/ a dGaussian/dx pulse

RANDOM Forcing $\Rightarrow$ shock structure: Fouque, Garnier & N., Physica D '04.

ASYMPTOTICS $\Rightarrow$ wave elevation $\equiv \eta(x,t)$ governed by **VISCOUS Burgers'**



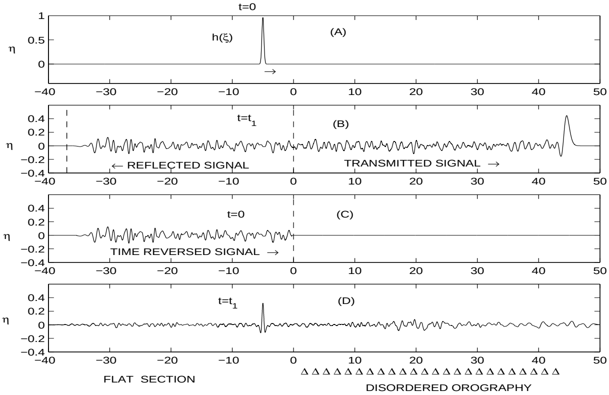
Time-reversal experiment: WAVEFORM INVERSION

Linear hyperbolic case:

Fouque & N. , SIAM MMS '04

$$\textbf{refocused pulse} \equiv \eta^{TR}(t) = \frac{1}{2\pi} \int e^{-i\omega t} \overline{\hat{\eta_0}}(\omega) \left(\frac{\alpha_m \omega^2 t_0'}{1 + \alpha_m \omega^2 t_0'} \right) d\omega .$$

$$\alpha_m = \int_0^{\infty} \mathbb{E}\left\{ m(0) m(x) dx \right\} \quad M(s) = 1 + m(s)$$

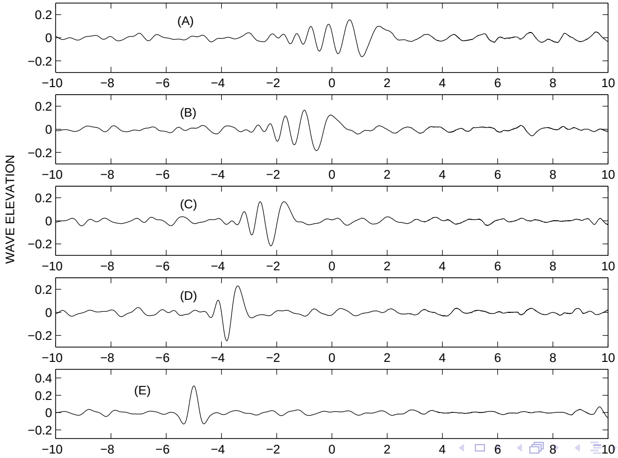


Linear Dispersive Gaussian: TIME-REVERSAL in REFLECTION

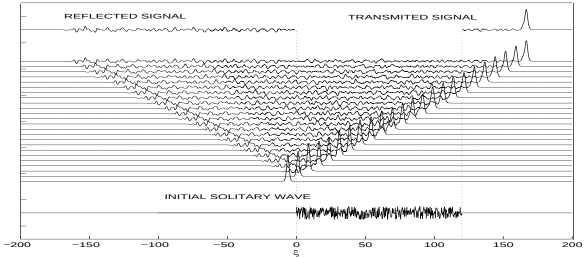
- Fouque, Garnier & N., SIAP '04:

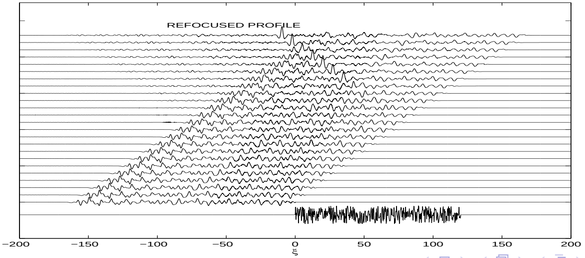
$$\eta_{trans}(t) \equiv \eta_0 * K\left(t - \sqrt{(\alpha_m/2)}B_L\right), \quad K \equiv K_{rand} * K_{disp}, \quad \alpha_m \equiv \int_0^\infty \mathbb{E}[m(0)m(s)]ds, \quad M(s) = 1 + m(s).$$

- Muñoz & N., SIAM MMS '05:



Solitary wave: Fouque, Garnier, Muñoz & N., PRL '04





Other members of the Boussinesq-family

(velocity monitored at other depth levels Z_0)

Let $Z_0 = \sqrt{2/3}$ and $u_\xi(\xi, t) = -M(\xi)\eta_t + O(\alpha, \beta)$:

$$(M(\xi)\eta)_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi - \frac{\beta}{6}(M(\xi)\eta)_{\xi\xi t} = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi - \frac{\beta}{6}u_{\xi\xi t} = 0$$

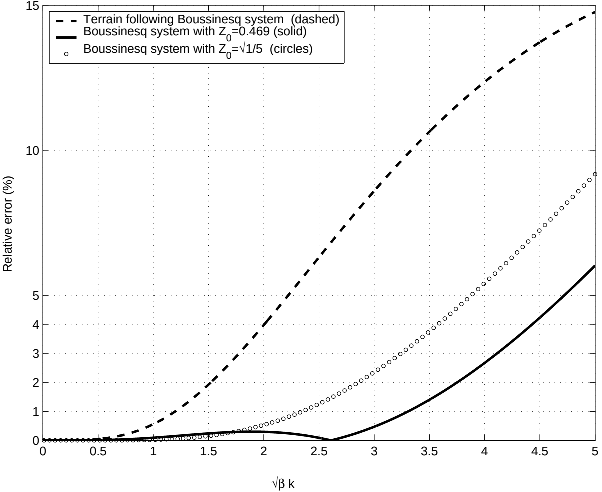
Quintero and Muñoz (Meth.Appl.Anal. '04) proved existence, uniqueness etc... by finding a conserved quantity. Main tool Bona & Chen '98

$$\left(\mathrm{I} - \frac{\beta}{6}\partial_{\xi\xi}\right)^{-1}[U] = K_\beta * U, \qquad K_\beta(s) \equiv -\frac{1}{2}\sqrt{\frac{6}{\beta}}sign(s)e^{-\sqrt{6/\beta}|s|}$$

$$E(t) \equiv \frac{1}{2}\int_{\Re}\left[\left(1 + \alpha\frac{\eta(\xi,t)}{M(\xi)}\right)[M(\xi)\eta(\xi,t)]^2 + M(\xi)\eta^2(\xi,t)\right]\ d\xi$$

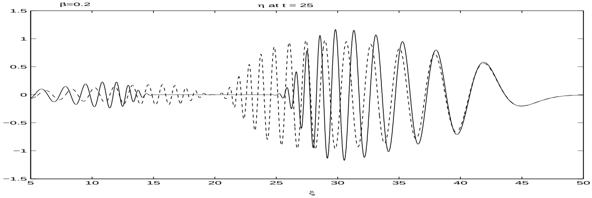
But more can be done!

Special values $Z_O = \sqrt{1/5} \approx 0.447$ and $Z_O = 0.469$: Muñoz & N., submitted '05

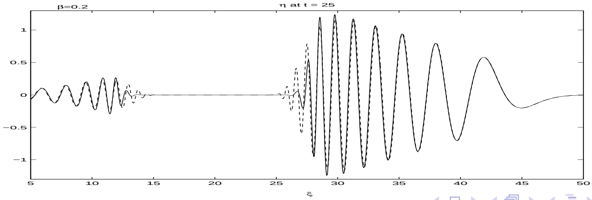


FLAT BOTTOM; solid line = linear potential theory

Depth-averaged $\Leftrightarrow \sqrt{1/3}$

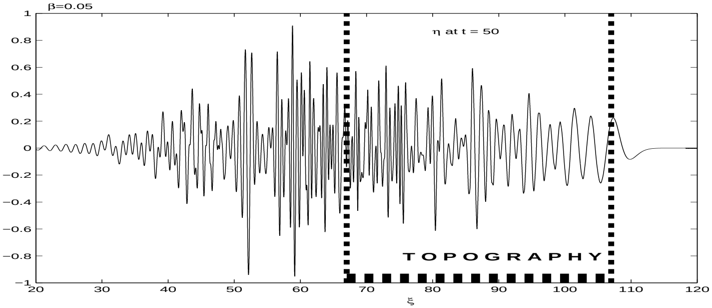


$Z_O = 0.469$ $\Leftarrow$ Nwogu '93



RAPIDLY VARYING TOPOGRAPHY

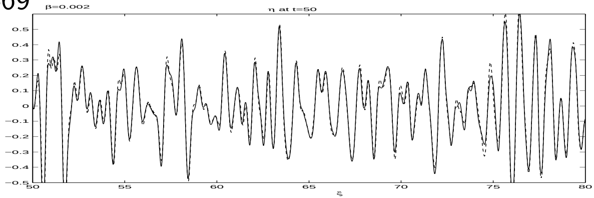
Will compare models HERE ⇓



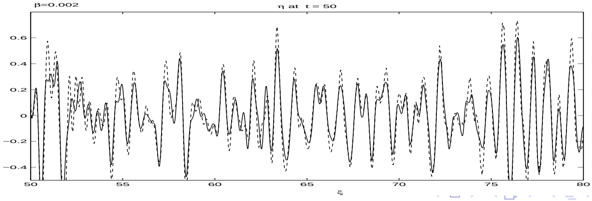
Energy transfer from WAVEFRONT to HIGHLY FLUCTUATING coda

DISORDERED TOPOGRAPHY: multiple-scattering signal

$Z_O = 0.469$



$Z_O = \sqrt{2/3}$ best value for Functional Analysis

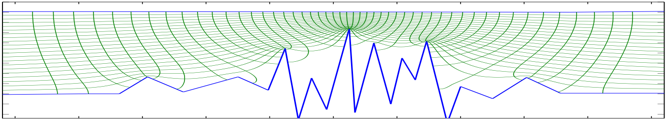


Can we do even better?

We had $\phi_{\xi\xi} + \phi_{\zeta\zeta} = 0, \qquad -\sqrt{\beta} < \zeta < S(\xi, t)$,
with a trivial Neumann condition at the bottom:

$$\phi_\zeta = 0 \quad \text{at} \quad \zeta = -\sqrt{\beta},$$

plus NONLINEAR free surface conditions.



NOW!! we solve the elliptic part **EXACTLY** (as opposed to the **POWER SERIES** expansion).
We construct the DIRICHLET-to-NEUMANN (DtN) map at the free surface.

The linear part $(S(\xi) \equiv 0) \Rightarrow \quad DtN_0[\varphi] \equiv \mathbf{T}[\varphi_\xi]$:

$$\mathbf{T}[\varphi_\xi] = -i \sum_{\kappa \neq 0} \tanh[2\pi\kappa\sqrt{\beta}]\mathbf{F}_\kappa[\varphi_\xi]e^{2\pi i\kappa\xi}.$$

Singular integral operator   Matsuno (PRL'92), Craig and Sulem (JCP'93):

Hilbert transform on the strip

$$\tilde{\mathbf{T}}[\varphi_x] = \frac{1}{2\sqrt{\beta}} \int_{-\infty}^{\infty} \frac{\varphi_x(x')}{\sinh[\frac{\pi}{2\sqrt{\beta}}(x - x')]} dx'.$$

Weakly nonlinear DtN operator: $S(\xi, t) \equiv \varepsilon \; N(\xi, t)$

$$|\Gamma| DtN[\varphi](\xi) = \mathbf{T}[\varphi_\xi] - \varepsilon\{\mathbf{T}[(N\,\mathbf{T}[\varphi_\xi])_\xi] - (N\varphi_\xi)_\xi\} + O(\varepsilon^2).$$

ITERATED linear DtN_0 **maps**

FULLY DISPERSIVE BOUSSINESQ SYSTEM:

over highly variable topography — Artiles & N., PhysRevLett '04

$$\eta_t - \frac{1}{M\sqrt{\beta}}\left\{\mathbf{T}[U] + \varepsilon\left(\left(\frac{\eta U}{M}\right)_\xi + \mathbf{T}\left[\frac{\eta}{M}\mathbf{T}[U]_\xi\right]\right)\right\} = O(\varepsilon^2)$$

$$U_t + \eta_\xi + \varepsilon\left(\frac{1}{2\sqrt{\beta}}\left(\left(\frac{U^2}{M^2}\right)_\xi + \left(\frac{1}{2M^2}\right)_\xi \mathbf{T}[U]^2\right) - \left(\frac{\eta}{M}\right)_\xi \mathbf{T}[\eta_\xi]\right) = O(\varepsilon^2)$$

$$\mathbf{T}[U] = -i\sum_{\kappa\neq 0}\tanh[2\pi\kappa\sqrt{\beta}]\mathbf{F}_\kappa[U]e^{2\pi i\kappa\xi}$$

- **Concluding remarks:**
- **FFT** (ODE-like scheme) $\Rightarrow$ **SPECTRAL ACCURACY in SPACE.**
- Existence of **SOLITARY WAVES**?
- Boussinesq-($Z_0 = 0.469$) **versus** Fully dispersive Boussinesq
  Explore in the presence of **heterogeneities**.

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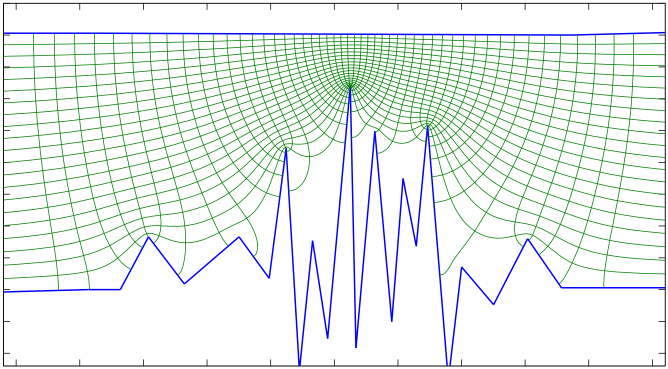
Math Modeling $\Rightarrow$ Preconditioning

Curvilinear coordinates preconditions the problem

$\Rightarrow$ smaller eigenvalue-band



Mild-slope dispersive model by Peregrine (1967) in (x,y):

$$\eta_t + [(h(x/\gamma)\ u)]_x = 0,$$
$$u_t + \eta_x + \beta\left(\frac{h^2(x/\gamma)}{6}u_{xxt} - \frac{h(x/\gamma)}{2}\partial_x^2(h(x/\gamma)\ u_t)\right) = 0.$$

Taking a Fourier transform in time we have that **($h_x \equiv h'/\gamma$)**

$$\hat{u}_x = -(\frac{h_x}{h})\ \hat{u} + (\frac{i\omega}{h})\ \hat{\eta},$$
$$\hat{\eta}_x = \frac{1}{1-\frac{\beta}{3}\omega^2 h}\left[i\omega\left(1-\frac{\beta}{6}hh_{xx}+\frac{\beta}{3}h_x^2\right)\hat{u}+\left(\frac{\beta}{3}\omega^2 h_x\right)\hat{\eta}\right],$$

The eigenvalues are

$$\lambda^{\pm}(x) = -\frac{h'}{\gamma}\left(1-\frac{1/2}{1-\frac{\beta}{3}\omega^2 h}\right) \pm \left[\frac{i\omega/h(x/\gamma)}{\sqrt{1-\omega^2\frac{\beta}{3}h(x/\gamma)}}\right]\times$$
$$\times\sqrt{1-\frac{\beta}{6\gamma^2}hh''+\frac{\beta}{3\gamma^2}h'^2-\frac{h'^2}{\gamma^2}\ \frac{(1-(2/3)\beta\omega^2 h)^2}{h(1-\frac{\beta}{3}\omega^2 h)}}.$$

As the scale ($\gamma \downarrow 0$)

$$\lambda^{\pm}(x) \sim O\left(\frac{1}{\gamma}\right) \pm i \left[\frac{\omega/h(x/\gamma)}{\sqrt{1-\omega^2\frac{\beta}{3}h(x/\gamma)}}\right] \cdot \sqrt{1 - O\left(\frac{1}{\gamma^2}\right)}.$$

Terrain-Following Boussinesq system by N. (2003) in (ξ,ζ):

$$M(\xi)\eta_t + u_\xi = 0,$$
$$u_t + \eta_\xi - \frac{\beta}{3}u_{\xi\xi t} = 0,$$

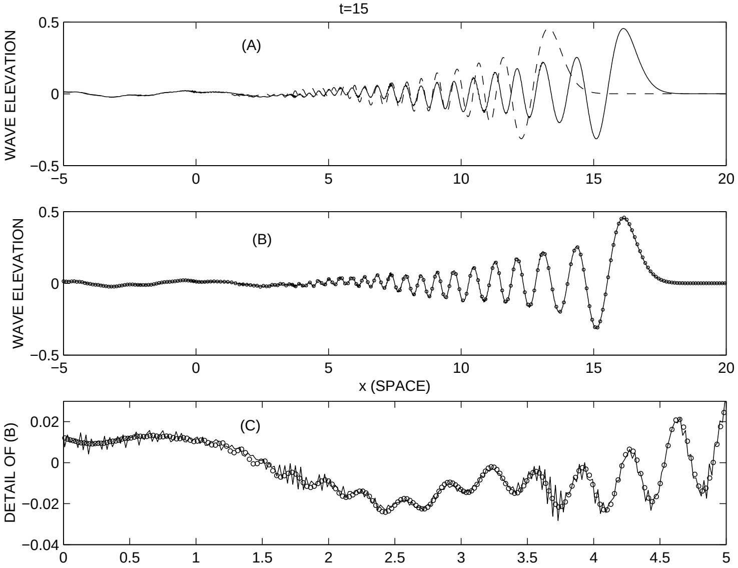
The eigenvalues are

$$\lambda^{\pm}(\xi) = \frac{\omega^2\frac{\beta}{3}M'(\xi) \pm i\omega\sqrt{M(\xi)}\sqrt{(1-\omega^2\frac{\beta}{3}M(\xi)) - \beta^2\frac{\omega^2 M'^2(\xi)}{36M(\xi)}}}{2(1-\omega^2\frac{\beta}{3}M(\xi))}.$$

With $\beta \ll 1 \Leftrightarrow$ long waves $\Rightarrow$ PRECONDITIONING Muñoz & N. SIAM MMS '05

$$\lambda^{\pm}(\xi) \sim O(\sqrt{\beta}) \pm i \left[\frac{\omega\sqrt{M(\xi;\beta,\gamma)}}{\sqrt{1-\omega^2\frac{\beta}{3}M(\xi;\beta,\gamma)}}\right] \cdot \sqrt{1+O(\beta)}.$$

Experiment with both models in the same regime:



Thank you for your attention.

IMPA, Rio de Janeiro, Brazil.