# The crack inverse problem

Dorin Bucur

LMAM UMR CNRS 7122, Metz
bucur @ math.univ-metz.fr

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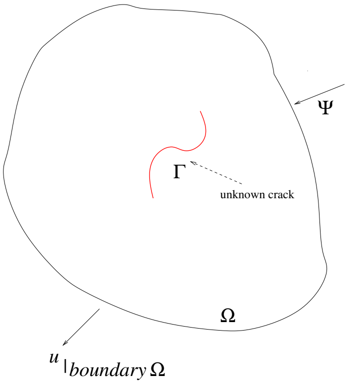
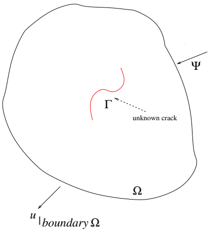





- Input the current flux $\Psi$
- Distribution of the current

$$\begin{cases} -\Delta u = & 0 \ \text{in} \quad \Omega \setminus \Gamma \\ \frac{\partial u}{\partial n} = & 0 \ \partial\Gamma \ \text{perfectly insulating} \\ \frac{\partial u}{\partial n} = & \Psi \ \ \partial\Omega \end{cases}$$

- Measure $u|_{\partial\Omega}$
- Problem: find $\Gamma$ if

$$\Psi \mapsto u|_{\partial\Omega}$$

is known.

# The main questions

1. Uniqueness: *may different cracks give the same measures*?
2. Stability: *close measures do give close cracks*?
3. Reconstruction: *give a geometric approximation of the position of the crack*.

# What is known?

- If the operator Neumann-to-Dirichlet is known the crack is uniquely defined
- One measure is not enough
- Two measures
  - 1989- Friedman and Vogelius: 1 smooth crack
  - 1992- Bryan and Vogelius: $N+1$ measures for $N$ smooth cracks
  - 1996- Kim and Seo: 2 measures for $N$ smooth cracks
  - 1996- Alessandrini and Diaz Valenzuela: 2 measures for $N$ non-smooth cracks or cavities

# Stability

All results require a priori uniform smoothness: equi-Lipschitz graphs

- 1989- Friedman and Vogelius

$$d_H(\gamma, \tilde{\gamma}) \leq C(\|u_1 - \tilde{u}_1\|_{L^\infty(\partial\Omega)} + \|u_2 - \tilde{u}_2\|_{L^\infty(\partial\Omega)})^\alpha$$

  $0 < \alpha \leq 1$ and $C$ depends on the Lip-norms of $\gamma, \tilde{\gamma}$

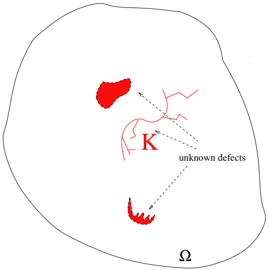
- 1992- Alessandrini $\longrightarrow$ 2004- Rondi: more results of the same type with better approximations

# Questions

- Uniqueness: characterize all sets which are identifiable by two boundary measurements.
  Are all sets identifiable quasi-everywhere?
- Stability: what happens if no a priori regularity is known for the crack?

Results obtained with:
Zakaria Belhachmi
Nicolas Varchon

# Uniqueness is related to the comprehension of the conjugate problem



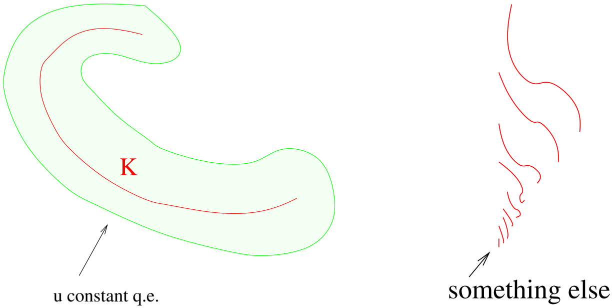

$$\begin{cases} -\Delta u = & 0 \text{ in } \Omega \setminus K \\ \frac{\partial u}{\partial n} = & 0 \ \partial K \\ \frac{\partial u}{\partial n} = & \Psi \ \partial \Omega \end{cases}$$

Does $u$ have an harmonic conjugate?

# What is the problem solved by the harmonic conjugate ?

Variational solution into

$$H^1_{cond,K}(\Omega) = cl_{H^1(\Omega)}\{u \in H^1(\Omega) : \exists \epsilon > 0,\ \nabla u = 0 \text{ a.e. on } K^\epsilon\}$$



# Conductive points

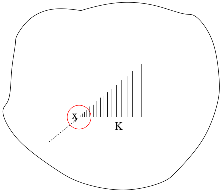
## Definition

$U$ open, $x_0 \in \partial U$ is called conductive if $\forall h \in C(\overline{U}) \cap H^1_{cond,\partial U \cap B(x_0,r)}(U)$ satisfies

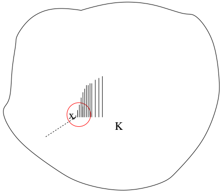
$$\liminf_{x \to x_0} \frac{|h(x) - h(x_0)|}{|x - x_0|} = 0.$$

Examples

- Continua are conductive at every point




$X$ is not conductive




$X$ is conductive

# The main uniqueness theorem

**Theorem**

*Let $K_1, K_2 \subset \Omega$ be q.e. conductive and $\Omega \setminus K_1$, $\Omega \setminus K_2$ connected. Two (well chosen) identical measures imply $K_1 = K_2$ q.e.*

**Example**

Sets $K$ such that $\forall x \in K \ \exists \ U_x$ continua of positive diameter, with $x \in U_x \subset K$ are conductive at every point.

Conjecture: Are all sets q.e. conductive?

# Stability

Hypothesis: $\sharp K \leq M$ (uniqueness holds)
Let $K_n, K$ such that

$$u_{K_n,\Psi_i}|_{\partial\Omega} \longrightarrow u_{K,\Psi_i}|_{\partial\Omega}, \;\; i = 1, 2$$

Is it true that

$$K_n \xrightarrow{H} K?$$

The Hausdorff convergence:

$$d_H(K_n, K) = |d(\cdot, K_n) - d(\cdot, K)|_\infty$$

Via the Tikhonov principle, stability relies on

- Stability of the direct problem (B.- Varchon 2000)
- Compactness and uniqueness

# The main stability result: $\sharp K \leq M$

**Theorem**

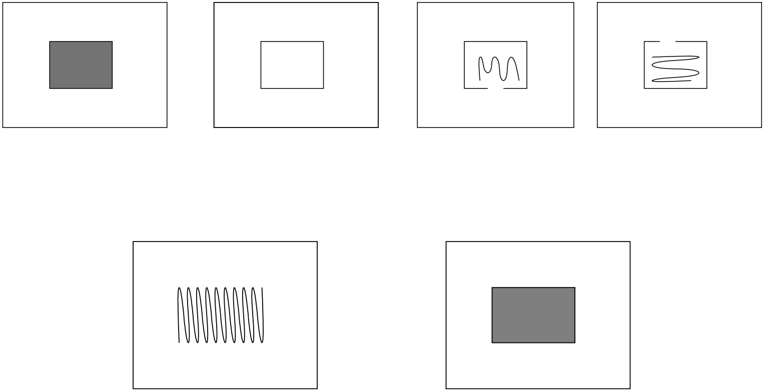
*Let $\psi_1, \psi_2$ well chosen. Assume*

$$u_{K_n,\psi_i|\partial\Omega} \xrightarrow{L^2(\partial\Omega)} u_{K,\psi_i|\partial\Omega} \quad i = 1, 2.$$

*For all convergent subsequences $K_{n_k} \xrightarrow{H} \tilde{K}$, we have*

$$co(\Omega \setminus K) = co(\Omega \setminus \tilde{K}) \quad q.e.$$

# Examples of stability



# Approximation by finite elements

Let $\mathcal{T}_h$ a discretization by finite elements.
Solve

$$\min_{\Gamma \subset \mathcal{T}_h, \sharp\Gamma \leq M} \int_{\partial\Omega} |u_{\Gamma,\Psi_1} - u_1|^2 + |u_{\Gamma,\Psi_2} - u_2|^2$$

$\Longrightarrow$ existence of a solution $\Gamma_h$

Question: $\Gamma_h \to \Gamma$?

Answer: YES, but... in the sense of the stability theorem