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OPTIMAL INTERNAL STABILIZATION OF A DAMPED WAVE EQUATION BY A LEVEL SET APPROACH

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INTRODUCTION - PROBLEMATIC

Let Ω be a Kirchhoff-Love plate and $\omega \subset \Omega$ be a piezo-electric device. $P(x,y) = \mathbb{1}_{((x,y)\in\omega)}$

Plate Model, clamped on Γ_0 and free on $\partial\Omega \backslash \Gamma_0$

$$
\begin{cases}
u - \textbf{transversal displacement of the plate} \\
\rho \dfrac{\partial^2 u}{\partial t^2} + D\Delta^2 u = -z_0^2 R \displaystyle\int_\omega \left(e_{31} \frac{\partial^3 u}{\partial x^2 \partial t} + e_{32} \frac{\partial^3 u}{\partial y^2 \partial t} \right) dx \times \left(e_{31} \frac{\partial^2 P}{\partial x^2} + e_{32} \frac{\partial^2 P}{\partial y^2} \right), \Omega \times (0,T) \\
+ \text{BOUNDARY CONDITIONS ON} \quad \partial\Omega \\
+ \text{INITIAL CONDITIONS AT} \quad t = 0
\end{cases}
\tag{1}
$$

$\Longrightarrow$ **Dissipative system.**

Problem to solve **Optimal position of the piezo ω in order to** MAXIMIZE THE DISSIPATION**.**

- $\Longrightarrow$ **Optimal shape design problem for time-dependent system**
- **Control of the evolution by the shape and position of the piezo.**

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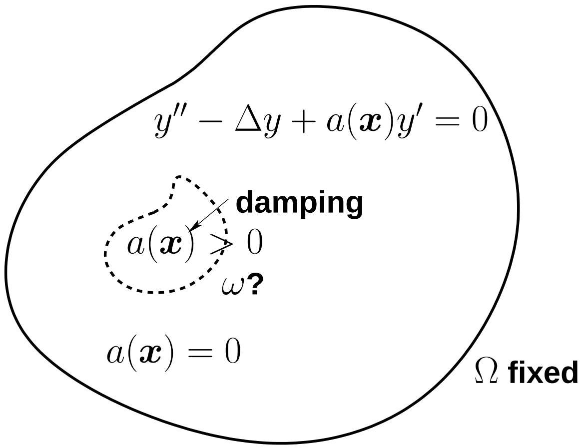
Simplification - Problem statement

Linear damped wave equation :

$$\begin{cases} y''_{\omega,a} - \Delta y_{\omega,a} + a(\boldsymbol{x}) y'_{\omega,a} = 0 & \text{in } \Omega \times (0,T), \\ y_{\omega,a} = 0 & \text{on } \partial\Omega \times (0,T), \\ y_{\omega,a}(.,0) = y_0, \quad y'_{\omega,a}(.,0) = y_1 & \text{in } \Omega. \end{cases} \quad (2)$$

where

- y_0, y_1**: initial position and velocity independent of** a **and** ω
- $\omega \subset \Omega$ **: dissipative zone (independent of** t**)**
- $a(x)$**: damping function; (usually,** $a(\boldsymbol{x}) = a\mathbb{1}_{(\boldsymbol{x}\in\omega)}$**)**



Remark 1 ***No condition on*** $\partial\omega$***. The system is well-posed (existence and uniqueness of the solution*** $y_{\omega,a}$***).***

REGULARITY

Definition 1 ***From*** $\Omega = (\Omega \backslash \omega) \cup \omega \cup \Gamma$

$$H^1(\Omega) = \{v \in L^2(\Omega), v_{|\omega} \in H^1(\omega), v_{|(\Omega\backslash\omega)} \in H^1(\Omega\backslash\omega), [[v]] = 0 \textbf{ on } \Gamma\}$$
$$H^2(\Omega) = \{v \in H^1(\Omega), v_{|\omega} \in H^2(\omega), v_{|(\Omega\backslash\omega)} \in H^2(\Omega\backslash\omega), [[\nabla v.\boldsymbol{\nu}]] = 0 \textbf{ on } \Gamma\} \quad \textbf{(3)}$$

Proposition 1 (Lions-Magenes, 1968) ***[Weak Solution] If*** $\Omega \in C^1(\mathbb{R})$ ***and*** $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ ***then***

$$y_{\omega,a} \in \mathcal{C}((0,T); H_0^1(\Omega)) \cap \mathcal{C}^1((0,T); L^2(\Omega)) \quad \textbf{(4)}$$

Proposition 2 (Regularity on $\partial\omega$) • $[[y_{\omega,a}]] = 0 \Longrightarrow [[\nabla y_{\omega,a}.\boldsymbol{\tau}]] = 0$

- $[[\nabla y_{\omega,a}.\boldsymbol{\nu}]] = 0$ ***(Regularizing effect of*** $-\Delta$***)***

Proposition 3 (Lions-Magenes) ***[Strong solution] If*** $\Omega \in C^2(\mathbb{R})$ ***et*** $(y_0, y_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ ***then***

$$y_{\omega,a} \in \mathcal{C}((0,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^1((0,T); H_0^1(\Omega)) \cap C^2((0,T); L^2(\Omega)) \quad \textbf{(5)}$$

Energy and Dissipation law

Definition 2 (Energy)

$$E(\omega,a,t)=\frac{1}{2}\int_\Omega \{|y'_{\omega,a}(\boldsymbol{x},t)|^2+|\nabla y_{\omega,a}(\boldsymbol{x},t)|^2\}dx,\quad \forall\ t\geq 0, \tag{6}$$

Proposition 4 (Dissipation law) ***Let*** $a(\boldsymbol{x})=a\mathbb{1}_{(\boldsymbol{x}\in\omega)}$***,***

$$E'(\omega,a,t)=-\int_\Omega a(\boldsymbol{x})|y'_{\omega,a}(\boldsymbol{x},t)|^2dx=-a\int_\omega |y'_{\omega,a}|^2dx\leq 0,\quad \forall t\geq 0. \tag{7}$$

For instance,

Theorem 1 (Haraux, Nakao, Zuazua, ...) ***[Exponential decay] If*** ω ***fulfills the Geometrical Optic Condition (GOC) :*** $\omega\in\mathcal{V}(\Gamma_0)\cap\Omega$***,*** $\Gamma_0=\{\,\boldsymbol{x}\in\partial\Omega\,;\,\boldsymbol{x}\cdot\boldsymbol{\nu}\ >0\,\}$ ***then***

$$\exists C,\alpha>0, E(\omega,a,t)\leq Ce^{-\alpha t}E(\omega,a,0),\quad \forall t\geq 0. \tag{8}$$

PROBLEMS STATEMENT

WHETHER OR NOT ω satisfies a geometrical condition, we consider the following NON LINEAR problems :

Problem 1 (Optimal position and shape design problem)

$$(\mathcal{P}_\omega) : \inf_{\omega \subset \Omega} E(\omega, a, T), \quad T > 0, a \in L^\infty(\Omega, \mathbb{R}^+) \tag{9}$$

Problem 2 (Optimization with respect to the function a)

$$(\mathcal{P}_a) : \inf_{a \in L^\infty(\Omega, \mathbb{R}^+)} E(\omega, a, T), \quad T > 0, \omega \subset \Omega \tag{10}$$

Problem 3 (Coupled problem !!)

$$(\mathcal{P}_{\omega,a}) : \inf_{\omega \subset \Omega, a \in L^\infty(\Omega, \mathbb{R}^+)} E(\omega, a, T), \quad T > 0 \tag{11}$$

$\Longrightarrow$ ω is assumed regular enough in order that $(\mathcal{P}_\omega)$ be well-posed.

Literature related to $(\mathcal{P}_a)$ and $(\mathcal{P}_\omega)$ - Over-damping phenomena

On a theoretical point of view, problems $(\mathcal{P}_a)$ and $(\mathcal{P}_\omega)$ are difficult because the energy is NOT MONOTONE with respect to a. It is the Over-Damping Phenomena

$$\forall a > 0, E(\omega, a, T) \leq E(\omega, 0, T) \quad \textbf{but} \quad \lim_{a \to \infty} E(\omega, a, T) = E(\omega, 0, T) \tag{12}$$

Literature is mainly concerned with the 1-D case (Stabilization of String) and $\omega = \Omega$ to the optimization of the exponential decay rate :

[Cox-Zuazua, 1994], [Freitas, 1998], [Lopez-Gomez, 1999],[Cox, Castro, 2001], ...

For example, in 1-D, with $\omega = \Omega = (0,1)$, the optimal constant value is $a = 2\pi$;

$$E(\omega, T, 2\pi) \leq E(\omega, a, T), \quad \forall a \geq 0 \tag{13}$$

while the optimal non constant function a is $a(x) = 2/x$ leading

$$E(\omega, 2/x, T) = 0, \quad \forall T > 2. \tag{14}$$

LITERATURE RELATED TO $(\mathcal{P}_\omega)$

FOR THE PROBLEM $(\mathcal{P}_\omega)$, NUMERICAL SIMULATION LEADS TO NON INTUITIVE RESULTS: THE OPTIMAL POSITION OF ω IS NOT SYMMETRICAL WITH RESPECT TO Ω. THE CAUSE IS THE OVER-DAMPING PHENOMENA

- **Hebrard P., Henrot A., *Optimal shape and position of the actuators for the stabilization of a string*, Systems and control letters, 48, 199-209 (2003).**
- **Henrot A., Maillot H., *Optimization of the shape and the location of the actuators in an internal control problem*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 3, 737-757 (2001).**
- **Hebrard P., *Etude de la géométrie optimale des zones de contrôle dans des problèmes de stabilisation*, PhD Thesis, Nancy, (2002).**

  [SIMULATION IN 2D USING GENETIC ALGORITHMS TO MAXIMIZE THE EXPONENTIAL DECAY RATE]

Not a Geometrical but an area constraint on ω

Conjecture 1 $\omega = \Omega$ ***is the trivial solution for Problem*** $(\mathcal{P}_\omega)$

Conjecture 2 $\omega_1 \subset \omega_2 \subset \Omega \Longrightarrow E(\omega_2, a, T) \leq E(\omega_1, a, T)$

[Open Questions !!]

Numerically, these two conjectures are observed. Consequently, an area constraint must be add on the dissipative zone ω

New Constrained Problem :

$$(\mathcal{P}_{\omega,\beta}) : \inf_{\omega \in V_\beta} E(\omega, a, T), \qquad V_\beta = \{\omega \subset \Omega, \textbf{area}(\omega) - \beta\textbf{area}(\Omega) = 0, \beta \in (0,1)\}$$

On the contrary, according to the nonlinearity of $a \longrightarrow E(\omega, a, T)$, no restriction is necessary on the damping function a

SHAPE DERIVATIVE OF E WITH RESPECT TO ω

Assume $\omega \in C^1(\Omega)$ **and let**

$$E^\varepsilon(\omega, a, T) \equiv E(\omega, a, T) + \frac{1}{2}\varepsilon^{-1}(\textbf{area}(\omega) - \beta\,\textbf{area}(\Omega))^2. \quad \textbf{(15)}$$

and $\boldsymbol{\theta} \in (W^{1,\infty}(\Omega, \mathbb{R}^2))^2$**,** $\boldsymbol{\theta}_{|\partial\Omega} = 0$**,** $\omega^\eta = (I + \eta\boldsymbol{\theta})(\omega) \equiv \mathcal{F}^\eta(\omega)$**.**

Theorem 2 ***The Fréchet derivative of*** E ***with respect to*** η ***(in the direction*** $\boldsymbol{\theta}$***) is***

$$\frac{\partial E^\varepsilon(\omega, a, T)}{\partial \omega}.\boldsymbol{\theta} = \int_{\partial\omega} \left[a \int_0^T y'(\boldsymbol{x}, t) p(\boldsymbol{x}, t) dt + \varepsilon^{-1}(\textbf{area}(\omega) - \beta\textbf{area}(\Omega)) \right] \boldsymbol{\theta}.\boldsymbol{\nu}\, d\sigma \quad \textbf{(16)}$$

$\boldsymbol{\nu}$***- normal oriented toward the exterior and*** p ***solution of the adjoint problem :***

$$\begin{cases} p''(\boldsymbol{x}, t) - \Delta p(\boldsymbol{x}, t) - a(\boldsymbol{x}) p'(\boldsymbol{x}, t) = 0 & \textit{in} \quad \Omega \times (0, T), \\ p(\boldsymbol{x}, t) = 0 & \textit{on} \quad (\partial\Omega \backslash \Gamma) \times (0, T), \\ p(\boldsymbol{x}, T) = -y'(\boldsymbol{x}, T) & \textit{in} \quad \Omega, \\ p'(\boldsymbol{x}, T) = -a(\boldsymbol{x}) y'(\boldsymbol{x}, T) - \Delta y(\boldsymbol{x}, T) & \textit{in} \quad \Omega. \end{cases} \quad \textbf{(17)}$$

$\boxed{\textbf{Descent direction } \boldsymbol{\theta} \textbf{ for } \omega \longrightarrow E^\varepsilon(\omega, a, T)}$ **(in order that** $E^\varepsilon(\omega + \eta\boldsymbol{\theta}(\omega), a, T) \leq E^\varepsilon(\omega, a, T)$**)**

$$\boldsymbol{\theta} = -\left(a \int_0^T y' p\, dt + \varepsilon^{-1}(\textbf{area}(\omega) - \beta\textbf{area}(\Omega)) \right) \boldsymbol{\nu} \quad \textbf{on} \quad \partial\omega \quad \textbf{(18)}$$

DERIVATIVE OF E WITH RESPECT TO $a(x)$

Let $a^\eta(\boldsymbol{x}) = a(\boldsymbol{x}) + \eta a^1(\boldsymbol{x})$**. Then,**

Theorem 3

$$\frac{\partial E(\omega, a, T)}{\partial a}.a^1 = \int_\omega \int_0^T a^1 y'_{\omega,a}(\boldsymbol{x}, t) p_{\omega,a}(\boldsymbol{x}, t)\, dt\, dx \tag{19}$$

$\boxed{\textbf{Descent direction for } a \longrightarrow E(\omega, a, T)}$

- **case** a **constant** $\Longrightarrow a^1 = -\int_\omega \int_0^T y' p\, dt\, dx \Longrightarrow \frac{\partial E(\omega,a,T)}{\partial a}.a^1 = -\left(\int_\omega \int_0^T y' p\, dt\, dx\right)^2 \leq 0$
- **case** a **non constant on** ω

  $\rightarrow a^1(\boldsymbol{x}) = -\int_0^T y' p\, dt \Longrightarrow \frac{\partial E(\omega,a,T)}{\partial a}.a^1(\boldsymbol{x}) = -\int_\omega \left(\int_0^T y' p\, dt\right)^2 dx \leq 0$

"TOPOLOGICAL" DERIVATIVE (ANALOGY)

[see Sokolowski (1999), Masmoudi (2001)]

$$a = 0 \Longrightarrow E(\omega, \eta a^1, T) = E(\omega, 0, T) + \eta a^1 \int_\omega \int_0^T y'_{\omega,0} p_{\omega,0} dt dx + o(\eta^2) \quad \textbf{(20)}$$

$y_{\omega,0}$ **- solution of the conservative case. For** $\omega = D(x_0, r) \subset \Omega$**,**

$$E(D(x_0,r), \eta a^1, T) = E(\emptyset, \eta a^1, T) + \eta a^1 \underbrace{\int_{D(x_0,r)} \int_0^T y'_{\omega,0} p_{\omega,0} dt dx}_{\equiv f(x_0)} + o(\eta^2), \quad \forall x_0 \in \Omega, \forall r > 0. \quad \textbf{(21)}$$

$\Longrightarrow$ **Useful to initialize the gradient algorithm:** $r > 0$ **fixed, minimize the function** $x_0 \to f(x_0)$**.**

USE OF THE LEVEL SET METHOD FOR THE PROBLEM $(\mathcal{P}_\omega)$

[Santosa (1996), Wang (2002), Allaire (2002), Burger (2003), Burger-Osher (2004) [a]]

$\Longrightarrow$ **Description of the moving boundary $\partial\omega$ INDEPENDENTLY of the representation of Ω, y and p, by the level set function ψ such that :**

$$\psi(\boldsymbol{x}) < 0 \quad \boldsymbol{x} \in \omega, \quad \psi(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \partial\omega, \quad \psi(\boldsymbol{x}) > 0 \quad \boldsymbol{x} \in \Omega\backslash\omega, \tag{22}$$

The evolving interface Γ is characterized by - $\tau > 0$ pseudo-time parameter increasing with time -

$$\partial\omega(\tau) = \{\boldsymbol{x}(\tau) \in \Omega, \psi(\boldsymbol{x}(\tau), \tau) = 0\} \tag{23}$$

After some manipulations, ψ solution of the ADVECTION TRANSPORT EQUATION

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} - j^{\varepsilon}(y_{\omega,a}, p_{\omega,a}, T)|\nabla\psi| = 0 \quad \textbf{in} \quad \Omega \times (0,\infty), \\ \psi(., \tau = 0) = \psi_0 \quad \textbf{in} \quad \Omega, \\ \psi = \psi_0 > 0 \quad \textbf{on} \quad \partial\Omega \times (0,\infty). \end{cases} \tag{24}$$

ensures the decrease of $\tau \rightarrow E^{\varepsilon}(\omega(\psi(\tau)), a, T)$. $\quad$ **j^{ε} is the integrand of the shape derivative.**

[a] M. Burger, S.J. Osher, A survey on level set methods for inverse problems and optimal design, European Journal of Mathematics, 2005

OPTIMIZATION ALGORITHM (FOR $\mathcal{P}_\omega$)

The GRADIENT DESCENT algorithm to solve numerically the problem $(\mathcal{P}_\omega)$ may be structured as follows :

1. MESHING ONCE FOR ALL **of the fixed domain Ω. Initialization of the level-set ψ_0 corresponding to an initial guess ω_0**

2. ITERATION **until convergence, for $k \geq 0$:**

   - COMPUTATION **on Ω of $y(\tau_k)$ (state problem) and $p(\tau_k)$ (adjoint problem).**

   - COMPUTATION **on Ω of the integrand $j_k^\varepsilon(y(\tau_k), p(\tau_k), T)$**

   - DEFORMATION **of the shape by solving the advection transport equation in ψ. The new domain $\omega(\tau_{k+1})$ is characterized by the level-set function $\psi(\tau_{k+1})$ after a pseudo-time step $\Delta\tau_k = \psi(\tau_{k+1}) - \psi(\tau_k)$ starting from the initial condition $\psi(\tau_k)$ with velocity $-j_k^\varepsilon$. The pseudo-time step $\Delta\tau_k$ is monitored by a stability condition.**

NUMERICAL RESOLUTION OF $y_{\omega,a}$ ET DE $p_{\omega,a}$

IN THE CONTEXT OF STABILIZATION, THERE IS A SERIOUS DIFFICULTY [a]

IN THE APPROXIMATION OF $y_{\omega,a}$, AN ADDITIONAL VISCOSITY TERM IS NECESSARY IN ORDER TO ENSURE THE CONVERGENCE OF THE ENERGY

$$y_h''(t) - \Delta_h y_h(t) + a_h y_t'(t) - h^2 \Delta_h y_h''(t) = 0 \quad \textbf{(25)}$$

PROBLEM COMES FROM THE SPURIOUS HIGH FREQUENCY SOLUTIONS (RELATED TO THE SPILL-OVER PHENOMENA)

Theorem 4 (M.-Pazoto in 2-D (2005)) ***If y_h is the solution of (25), if Ω is C^2, and $E_h(0) \to E(0)$, then the discrete energy $E_h(t)$ is such that***

$$E_h(t) \to E(t), \quad \textit{when} \quad h \to 0 \quad \textbf{(26)}$$

In particular, EXPONENTIAL DECAY PROPERTY ***and*** NUMERICAL APPROXIMATION ***commute.***

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[a] **Zuazua-Tebou (2003), Munch,-Pazoto (2005), Tucsnak (2005), ...**

Loss of uniform convergence - One 2-D example

Irregular initial conditions (M., Pazoto, 2005):

$$y_0(\boldsymbol{x}) = \textbf{sin}(-\pi x_1 \frac{1}{h})\textbf{sin}(-\pi x_2 \frac{1}{h}) - \textbf{sin}(\pi x_1 (1 - \frac{1}{h}))\textbf{sin}(\pi x_2 (1 - \frac{1}{h})) \quad ; \quad y_1(\boldsymbol{x}) = 0. \qquad \textbf{(27)}$$

- **Without damping function, $E_h(t)$ is constant**
- **Without viscous terms, $E_h(t) \nrightarrow E(t)$ (the dissipation is lost)**
- **With viscous terms, $E_h(t) \rightarrow E(t)$ (the dissipation is restored)**

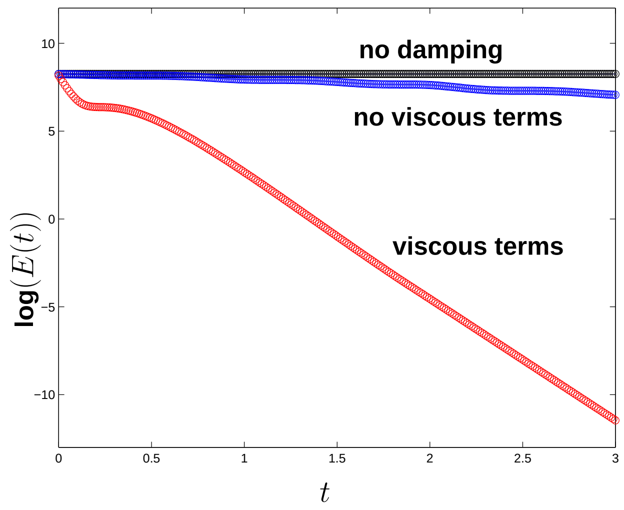


Figure 1: $\log(E(t))$ vs. t; $h = 1/62$.

NUMERICAL APPLICATIONS- $(\mathcal{P}_\omega)$ - Ω = UNIT SQUARE

$$y_0(\boldsymbol{x}) = 100\sin(\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a(\boldsymbol{x}) = a\,1_\omega(\boldsymbol{x}) \tag{28}$$

FOR a SMALL ENOUGH, AN ANALYTICAL CALCULATION LEADS TO

$$E(\omega, a, T) - E(\omega, 0, T) = -\frac{a\sqrt{2}\pi}{4}\left(2\alpha T - \sin(2\alpha T)\right)\int_\omega y_0^2 dx + O(a^2) \tag{29}$$

AND SHOWS THAT THE OPTIMAL POSITION OF ω CORRESPONDS TO THE MAXIMUM OF y_0^2, I.E. AT $(1/2, 1/2)$

NUMERICAL EXAMPLE - $(\mathcal{P}_\omega)$

$$y_0(\boldsymbol{x}) = 100\sin(\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a = 10. \qquad \textbf{(30)}$$

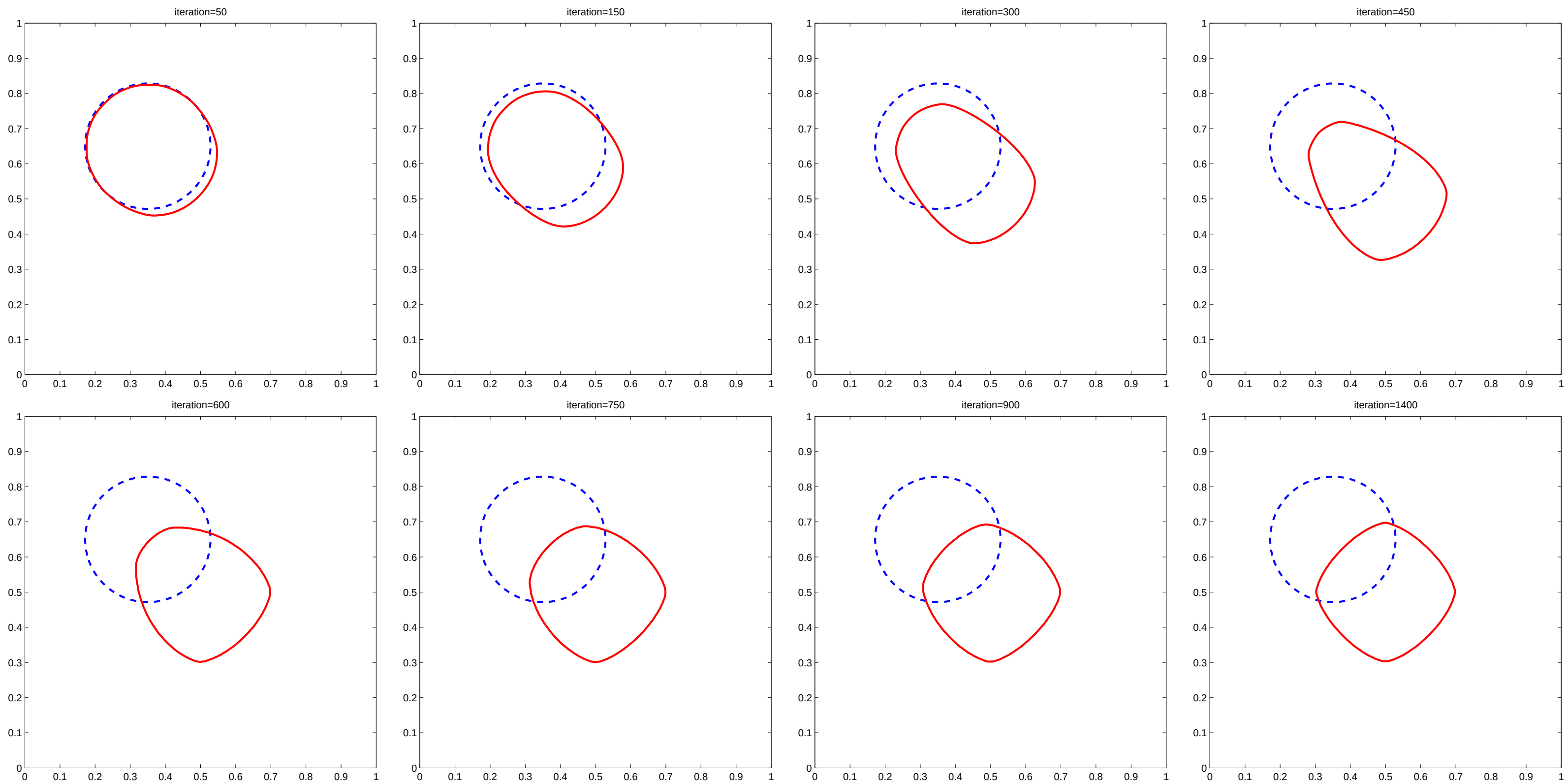
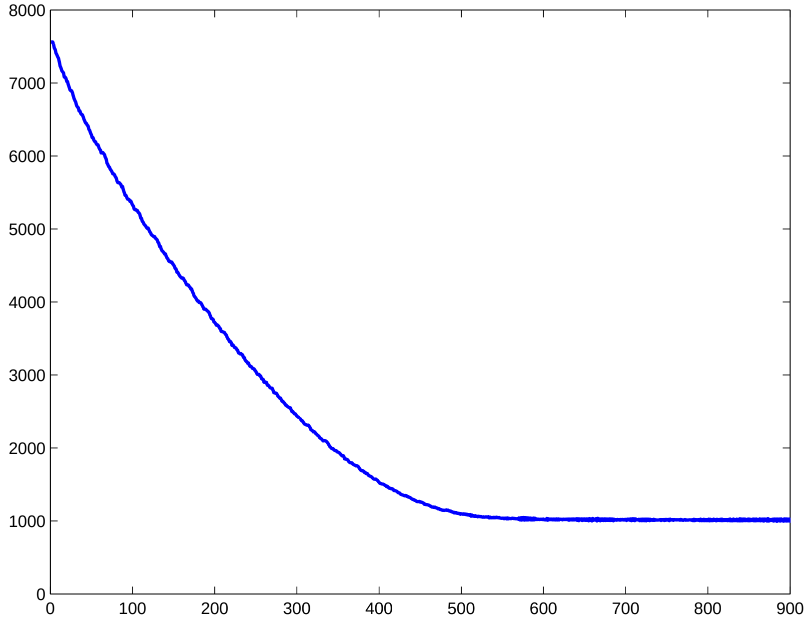


Figure 2: $T = 1.$, $a = 10.$ **-** $\{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ **vs.** k

NUMERICAL EXAMPLE - $(\mathcal{P}_\omega)$

$$y_0(\boldsymbol{x}) = 100\sin(\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a = 10. \tag{31}$$



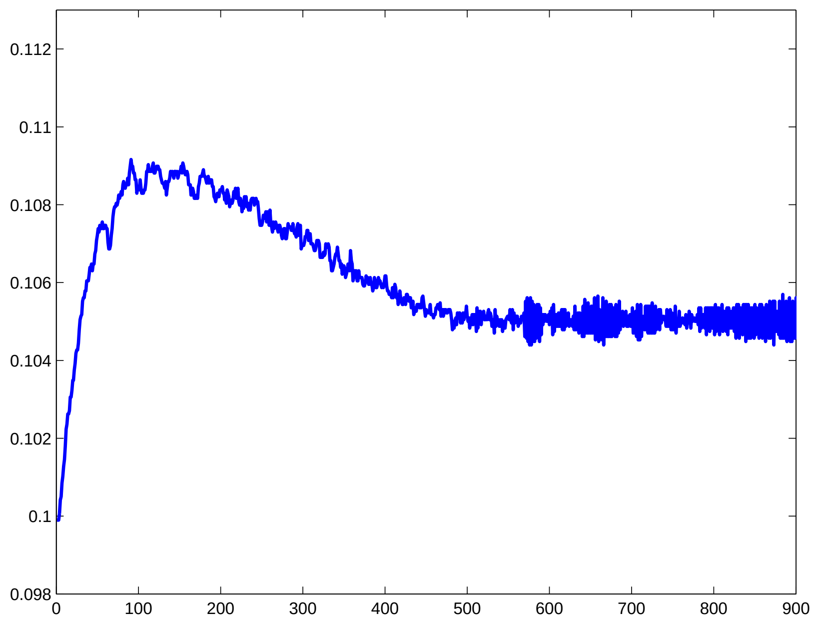


Figure 3: $T = 1$ **- Evolution of** $E(\omega_k, a, T)$ **(left) and area(**ω_k**)(right) vs.** k **-** $h = 1/151$**.**

$area(\omega_k) \approx 0.1$

NUMERICAL EXAMPLE - $(\mathcal{P}_\omega)$

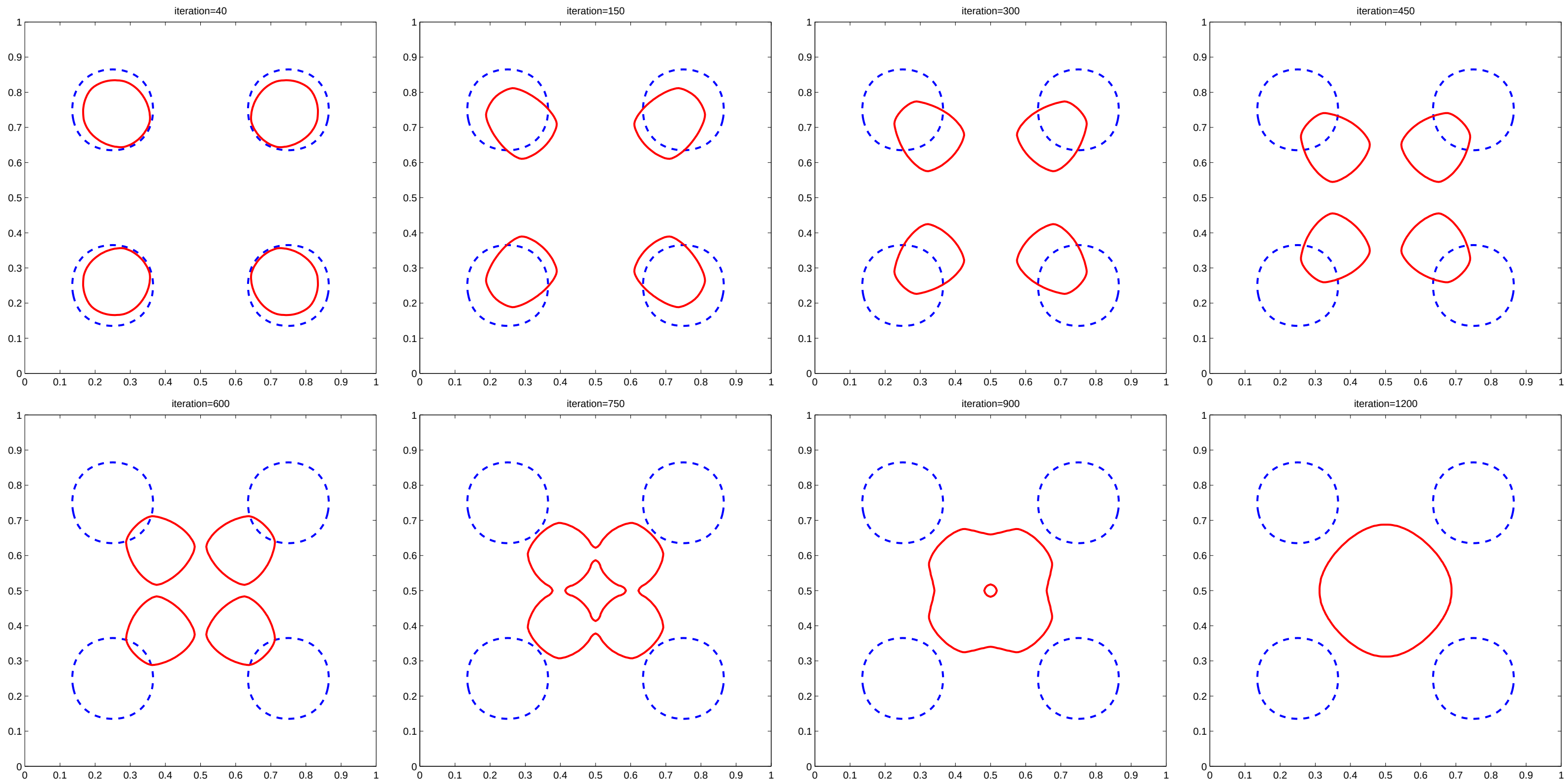


Figure 4: ω_0 composed of four parts : invariance.

Numerical example - $(\mathcal{P}_\omega)$

$$y_0(\boldsymbol{x}) = 100\sin(\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a = 10. \tag{32}$$

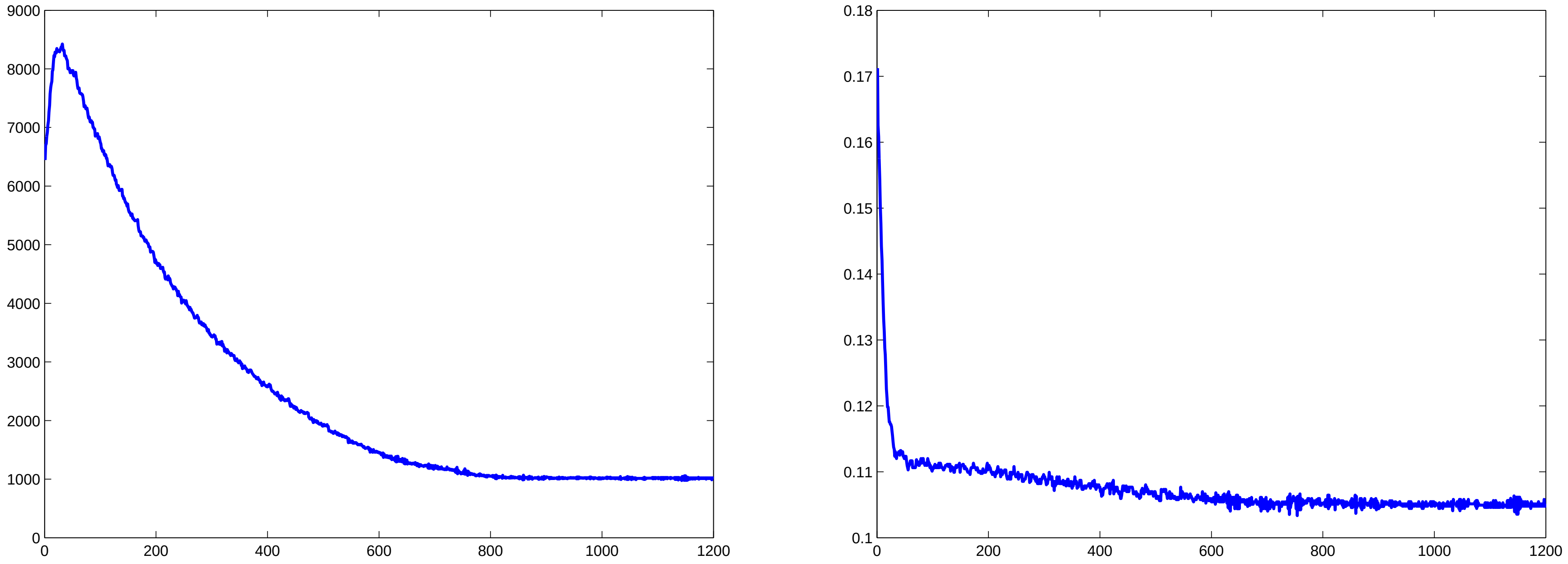


Figure 5: $T = 1$ **- Evolution of** $E(\omega_k, a, T)$ **(left) and area(**ω_k**)(right) vs.** k **-** $h = 1/151$**.**

$$y_0(\boldsymbol{x}) = 100\sin(2\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0 \tag{33}$$

ONCE AGAIN, FOR a SMALL ENOUGH, THE OPTIMAL POSITION IS RELATED TO THE MAXIMA OF y_0^2. CONSEQUENTLY, THE OPTIMAL POSITION IS COMPOSED OF TWO PARTS CENTERED ON $(1/4, 1/2)$ AND $(3/4, 1/2)$.

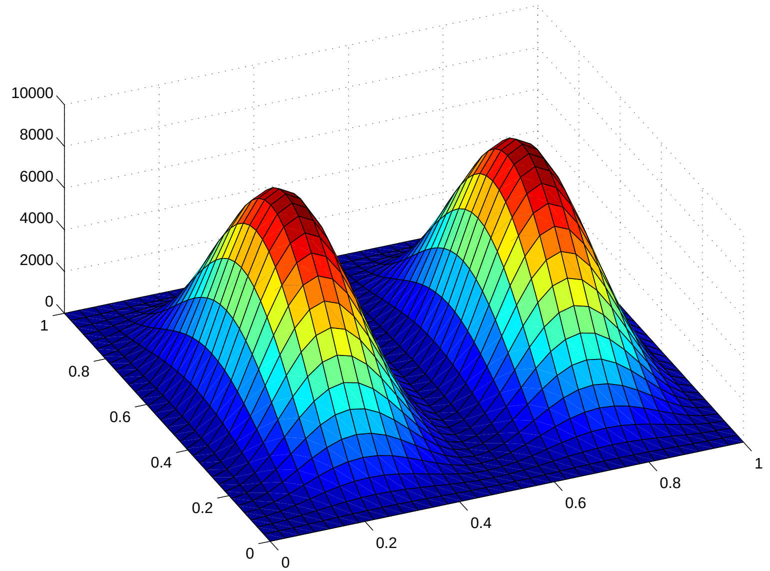


Figure 6: $(y_0(\boldsymbol{x}))^2$

$$E(\omega, a, T) - E(\omega, 0, T) = \frac{-a\sqrt{5}\pi}{4}\left(2\sqrt{5}\pi T - \sin(2\sqrt{5}\pi T)\right)\int_\omega (y_0(\boldsymbol{x}))^2 dx + o(a^2), \quad \forall T > 0, \tag{34}$$

NUMERICAL EXAMPLE - $(\mathcal{P}_\omega)$

$$y_0(\boldsymbol{x}) = 100\sin(2\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a = 10. \tag{35}$$

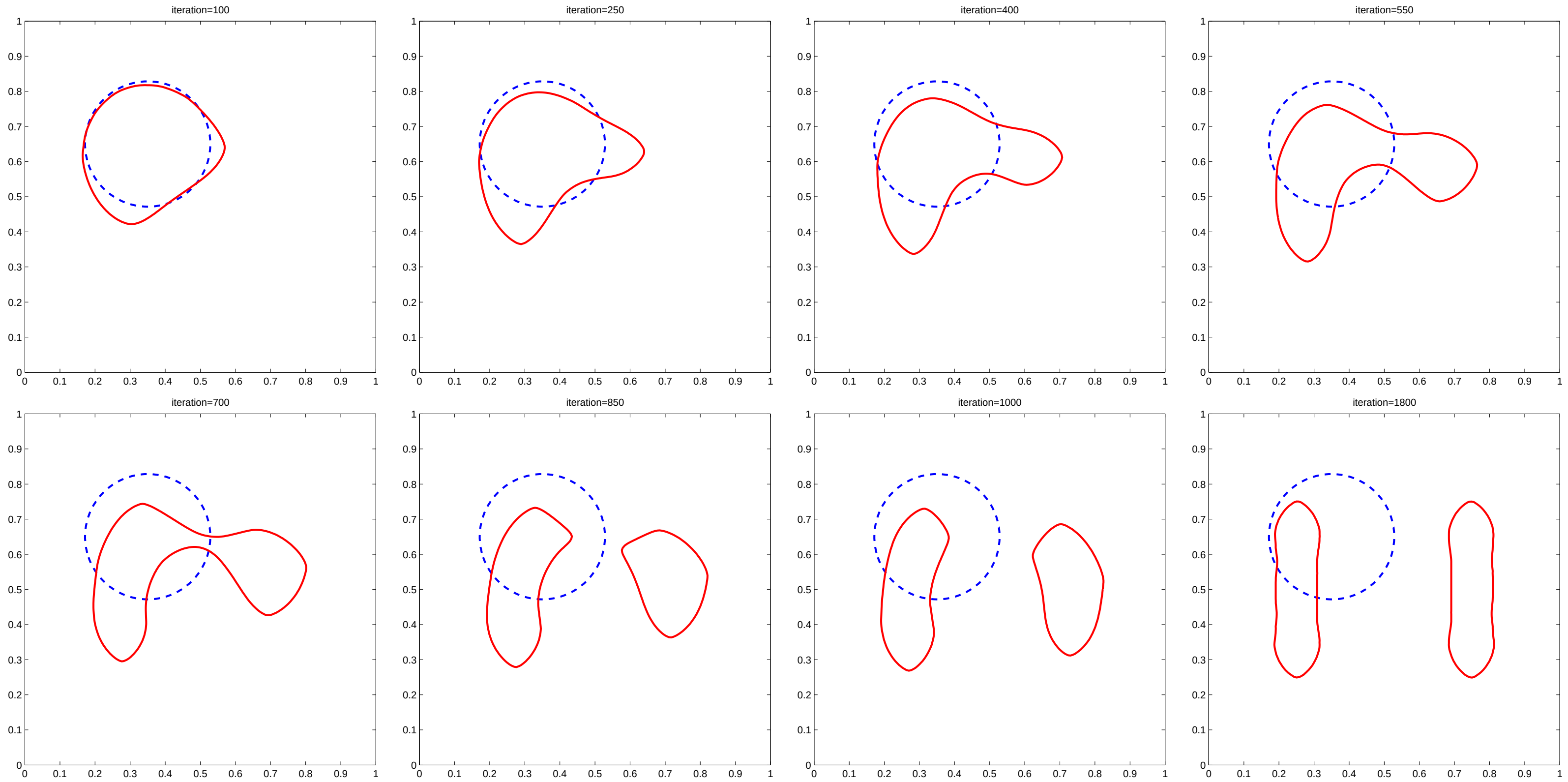


Figure 7: $T = 1.$, $a = 10.$ - $\{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ **vs.** k

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NUMERICAL EXAMPLE - $(\mathcal{P}_\omega)$

$$y_0(\boldsymbol{x}) = 100\sin(2\pi x_1)\sin(\pi x_2), \quad y_1(\boldsymbol{x}) = 0, \boldsymbol{x} = (x_1, x_2) \in (0,1)^2, \quad a = 10. \tag{36}$$

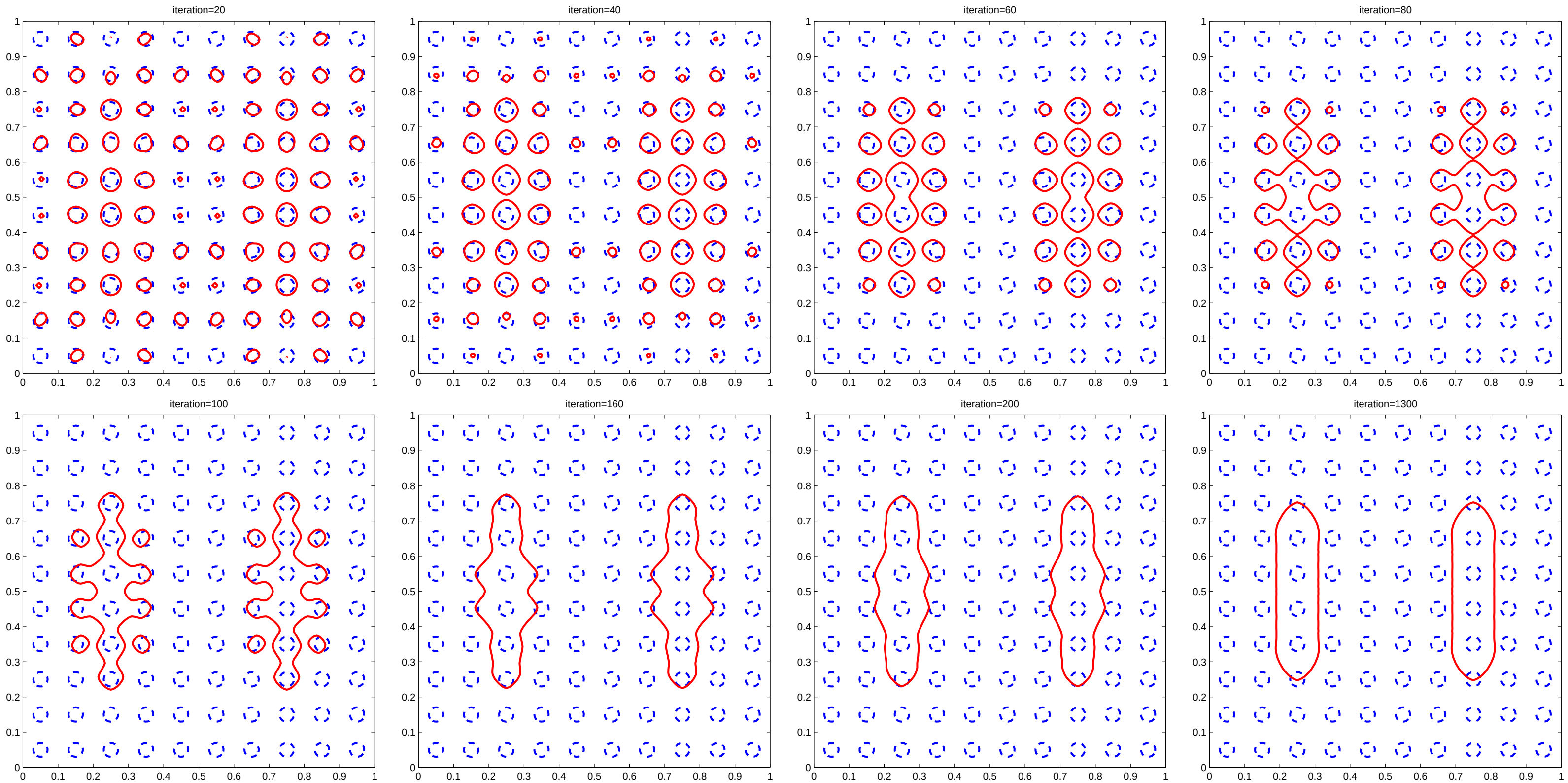

Figure 8: $T = 1., a = 10.$ - $\{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ **vs.** k

Numerical Example - $(\mathcal{P}_\omega)$ - Dependence in T

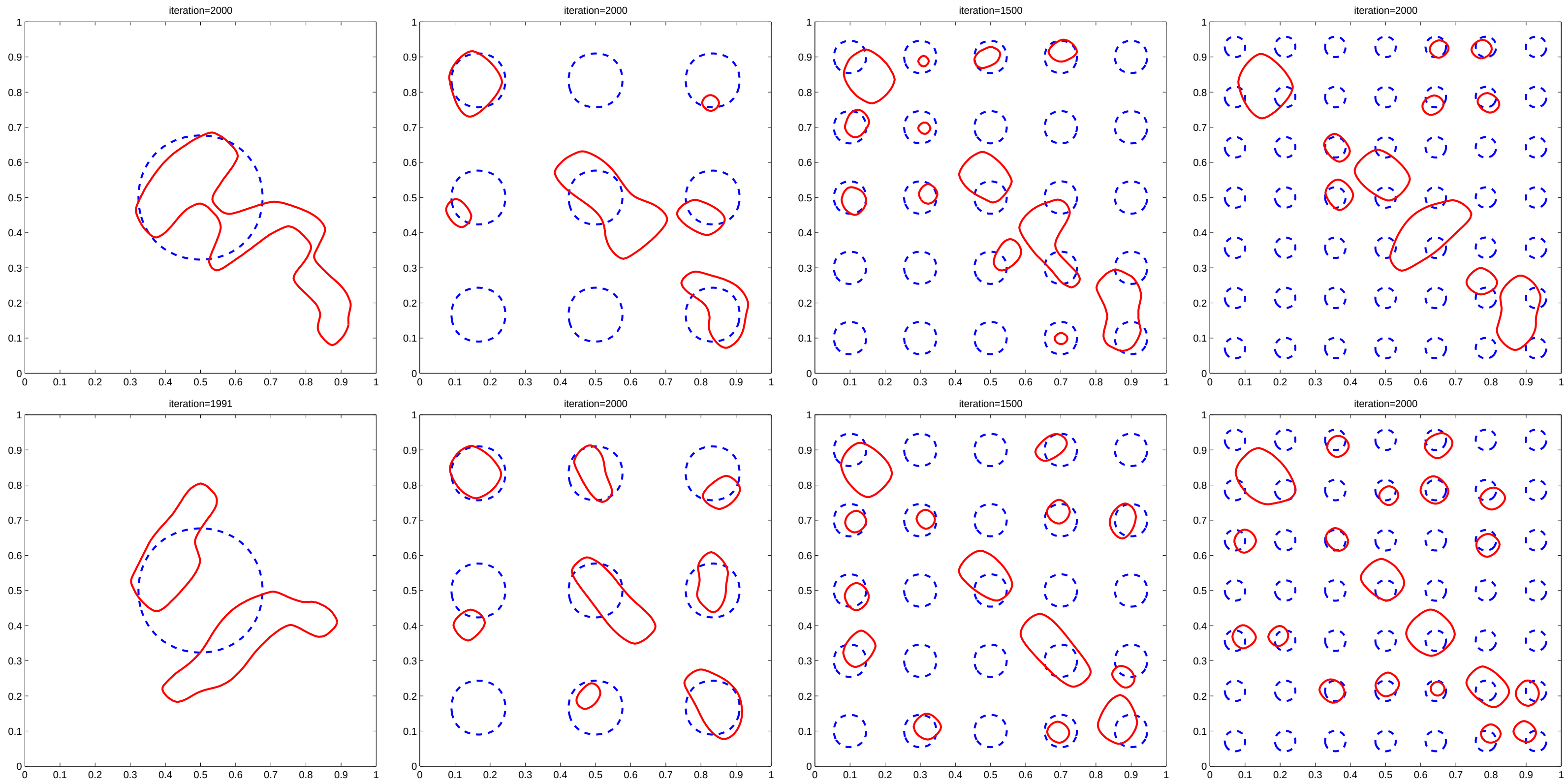

Figure 9: $(y_0, y_1) = (300x_1x_2(x_1-1)(x_2-1)\cos(5\pi x_1(x_2-1))\sin(2\pi x_1x_2), 0)$, $a = 10$. **- "limit" of** $\{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ **for several** ω_0**. -** $T = 1$ **(Top) et** $T = 2$ **(Bottom).**

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$(\mathcal{P}_\omega)$ - Dependence with respect to T and ω_0 - Energy

T	$\sharp\,\omega_0 = 1$	$\sharp\,\omega_0 = 9$	$\sharp\,\omega_0 = 25$	$\sharp\,\omega_0 = 49$
1	502.64	261.88	256.86	249.10
2	322.88	117.99	96.53	88.17

Table 1: Value of the cost function $E(\omega, a = 10., T)$ for different initialization ω_0 - $\sharp\,\omega_0$: number of disjoint parts of ω_0

Remark 2

- ***The minimum is obtained for ω_0 composed of the highest number of disjoint parts.***
- ***Dependence with respect to the initialization ω_0 due to the multiplicity of the local minima.***

PROBLEM $(\mathcal{P}_a)$ - $\omega = \text{DISC}(r = \sqrt{0.1/\pi}, (1/2, 1/2))$

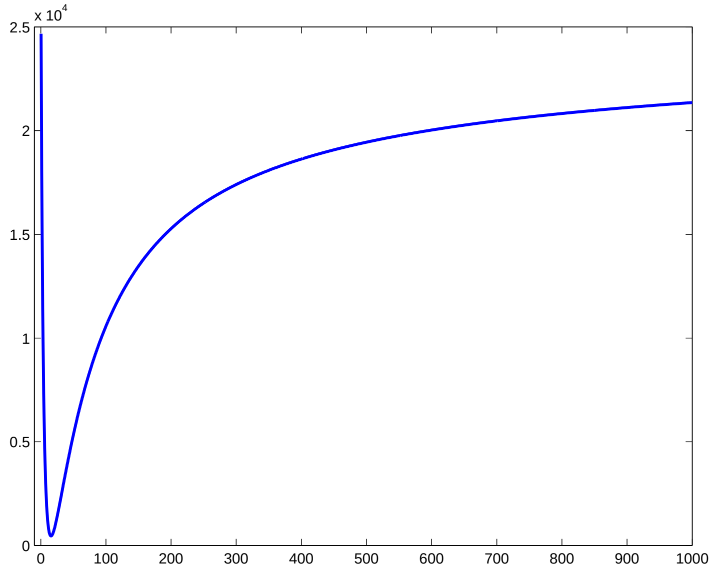


Figure 10: Over-damping phenomena - $E(\omega, a, T = 1)$ **vs.** a

- $E(\omega, 0, T) - E(\omega, a, T) \approx O(a^{-0.78})$, $a >> 1$.

  $a_{opt} \approx 15.338$ - $E(\omega, a_{opt}, 1) \approx 439.592$

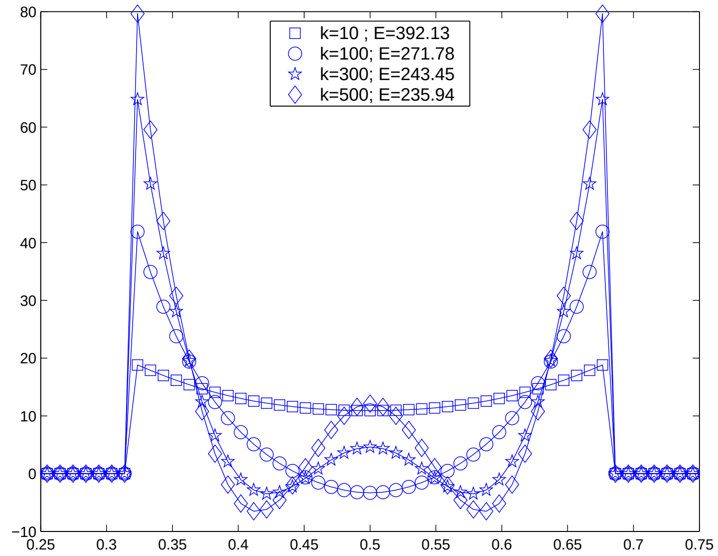
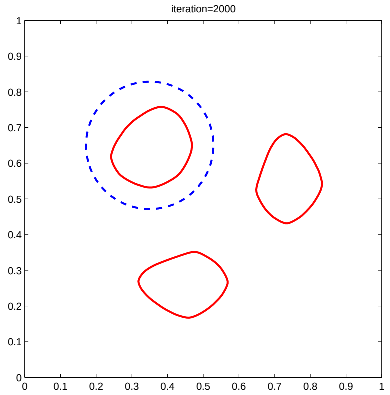


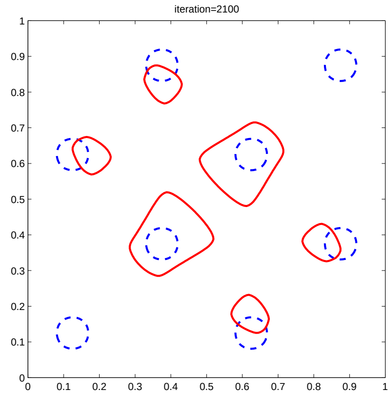
Figure 11: Optimization of $a(\boldsymbol{x})$ - $a_k(x_1, x_2 = 1/2)$ **vs.** x_1

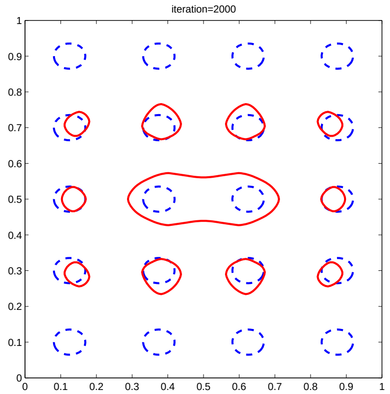
Remark 3 $\int_\omega a_k(\boldsymbol{x})dx / \int_\omega dx \to a_{opt}$

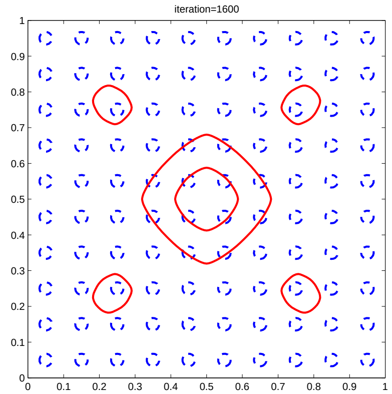
Remark 4 ***For*** $\omega =$ ***disc***$(r = \sqrt{0.1/\pi}, (0.35, 0.35))$, $a_{opt} = 24.89$ ***and*** $E(\omega, a_{opt}, T = 1) = 4916.93$

BACK ON PROBLEM $(\mathcal{P}_\omega)$ WITH $a = 24.89$ (INSTEAD OF $a = 10.$)









$\sharp\,\omega_0 = 1$	$\sharp\,\omega_0 = 8$	$\sharp\,\omega_0 = 20$	$\sharp\,\omega_0 = 100$
101.08	93.47	34.82	50.35

Problem $(\mathcal{P}_{\omega,a})$

$$(y_0, y_1) = (100 \sin(\pi x_1) \sin(\pi x_2), 0), \quad T = 1 \tag{37}$$

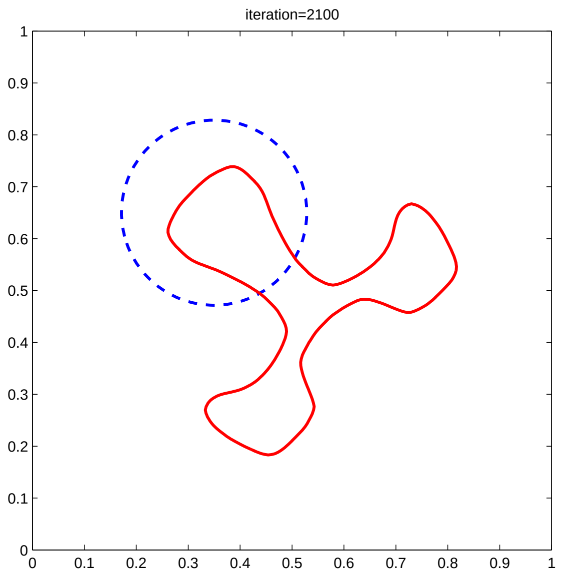


Figure 12: $\lim \partial\omega_k$

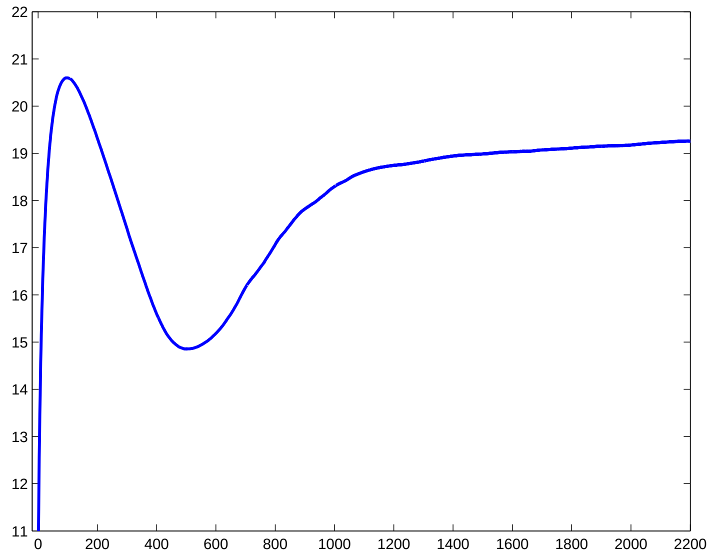


Figure 13: a_k **(constant on** ω_k**) vs. iteration** k

$$a_{2000} \approx 19.51, \quad E(\omega_{2000}, a_{2000}, T = 1) \approx 140.12 \tag{38}$$

PROBLEM $(\mathcal{P}_{\omega,a})$

$$(y_0, y_1) = (100 \sin(\pi x_1) \sin(\pi x_2), 0), \quad T = 1 \tag{39}$$

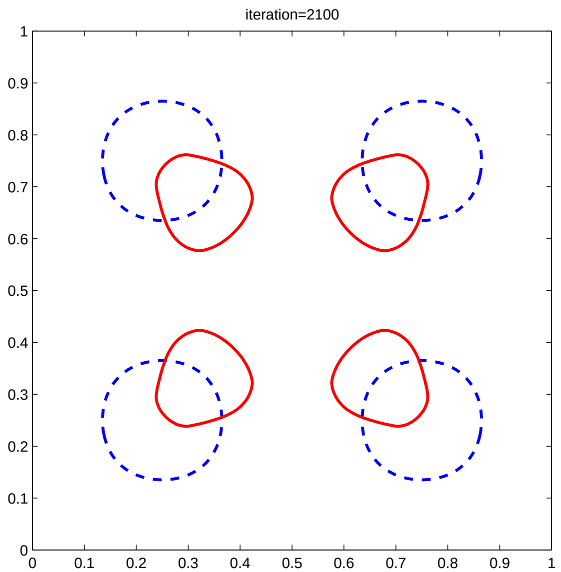


Figure 14: $\lim \partial\omega_k$

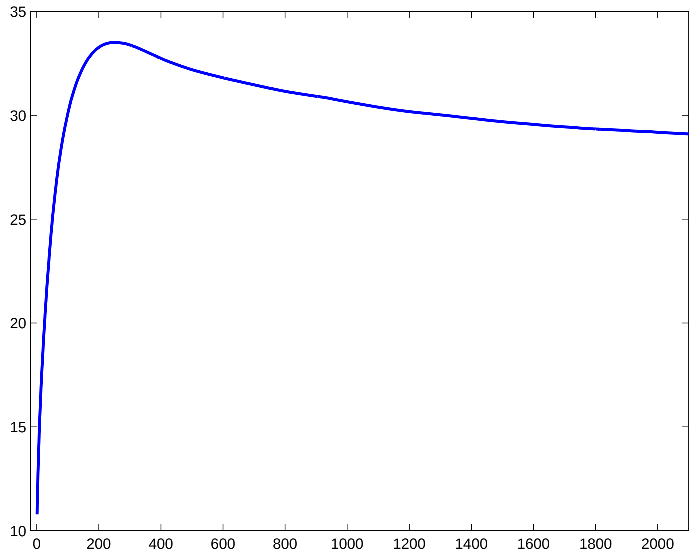
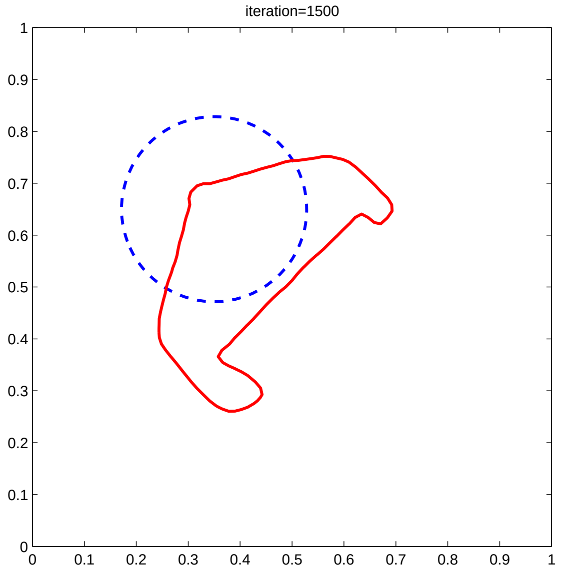


Figure 15: a_k **(constant on** ω_k**) vs. iteration** k

$$a_{2000} \approx 29.098, \quad E(\omega_{2000}, a_{2000}, T = 1) \approx 12.41 \tag{40}$$

Problem $(\mathcal{P}_\omega)$ - Singular initial conditions

$$y_0(\boldsymbol{x}) = \begin{cases} 40 & (x_1, x_2) \in (\frac{1}{3}, \frac{2}{3})^2 \\ 0 & \textbf{elsewhere} \end{cases} \quad ; \quad y_1(\boldsymbol{x}) = 0; \qquad \textbf{(41)}$$



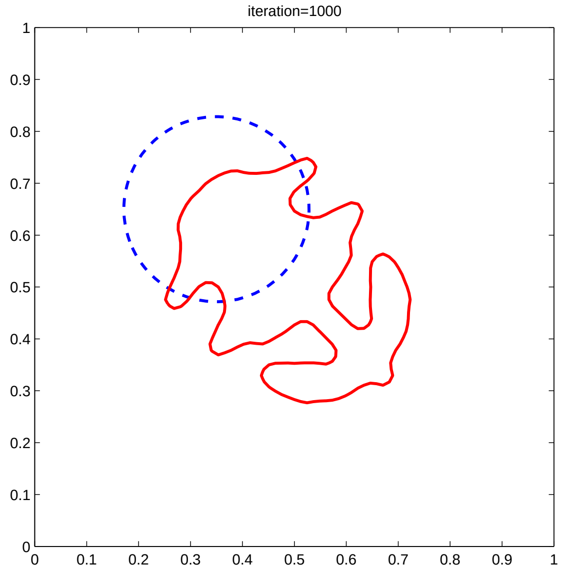


Figure 16: Singular case. $T = 1$, $a = 24.89$ **- "limit" in** k **of the zero-level set sequence** $\{\boldsymbol{x} \in \Omega, \psi_k(\boldsymbol{x}) = 0\}$ **- left: without viscosity terms** $E(\omega, a, T) \approx 2768.70$ **- right: with viscosity terms** $E(\omega, a, T) = 1487.23$**.**

Boundary stabilization

$$\begin{cases} y''_{\gamma,a}(\boldsymbol{x},t) - \Delta y_{\gamma,a}(\boldsymbol{x},t) = 0 & \textbf{in } Q = \Omega \times (0,T), \\ y_{\gamma,a}(\boldsymbol{x},t) = 0 & \textbf{on } \gamma_0 \times (0,T), \\ \nabla y_{\gamma,a}(\boldsymbol{x},t).\boldsymbol{\nu} = 0 & \textbf{on } \gamma_1 \times (0,T), \\ \nabla y_{\gamma,a}(\boldsymbol{x},t).\boldsymbol{\nu} = -a(\boldsymbol{x})y'(\boldsymbol{x},t) & \textbf{on } \gamma \times (0,T), \\ y_{\gamma,a}(\boldsymbol{x},0) = y_0(\boldsymbol{x}), \quad y'_{\gamma,a}(\boldsymbol{x},0) = y_1(\boldsymbol{x}) & \textbf{in } \Omega, \end{cases} \quad \textbf{(42)}$$

Proposition 5 (Dissipation law) $a(\boldsymbol{x}) = a\, 1_\gamma(\boldsymbol{x})$**,** $a > 0$**;**

$$E'(\gamma,a,t) = -\int_{\partial\Omega} a(\boldsymbol{x}) |y'_{\gamma,a}(\boldsymbol{x},t)|^2 d\sigma = -a \int_\gamma |y'_{\gamma,a}|^2 d\sigma \leq 0, \quad \forall t \geq 0. \quad \textbf{(43)}$$

Problem 4 $(\mathcal{P}_\gamma)$***: Optimal position of*** $\gamma \subset \partial\Omega \backslash \gamma_0$ ***minimizing*** $E(\gamma,a,T)$**.**

PROBLEM $(\mathcal{P}_\gamma)$ - BOUNDARY CASE

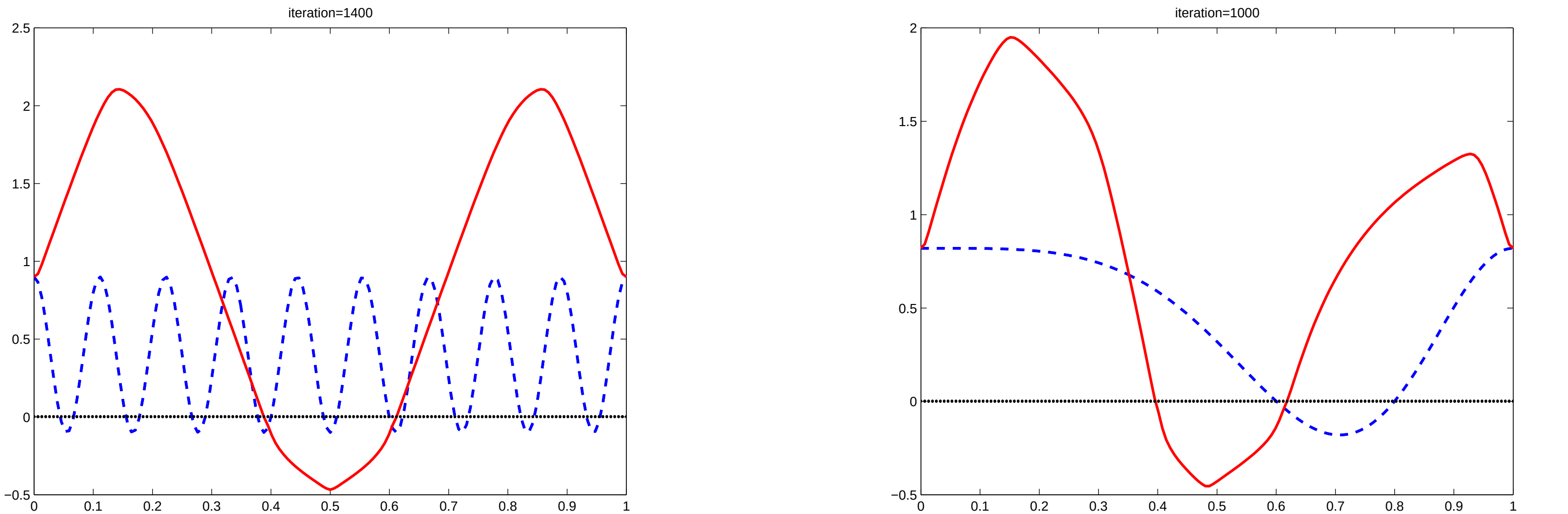


Figure 17: $(y_0, y_1) = (130 \sin(\pi x_1) \sin(\pi x_2/2), 0)$, $T = 2$, $a = 1$. **- Limit of** ψ_k **on** $\gamma \subset \partial\Omega$ **-** $h = 1/151$ **for different initialization** ψ_0. **Top left:** $\psi_0(x_1) = 0.9 - \sin(9\pi x)^2$**- Top right :** $\psi_0(x_1) = 0.9 - \sin(\pi x^2)^2$

PROBLEM $(\mathcal{P}_\gamma)$ - BOUNDARY CASE

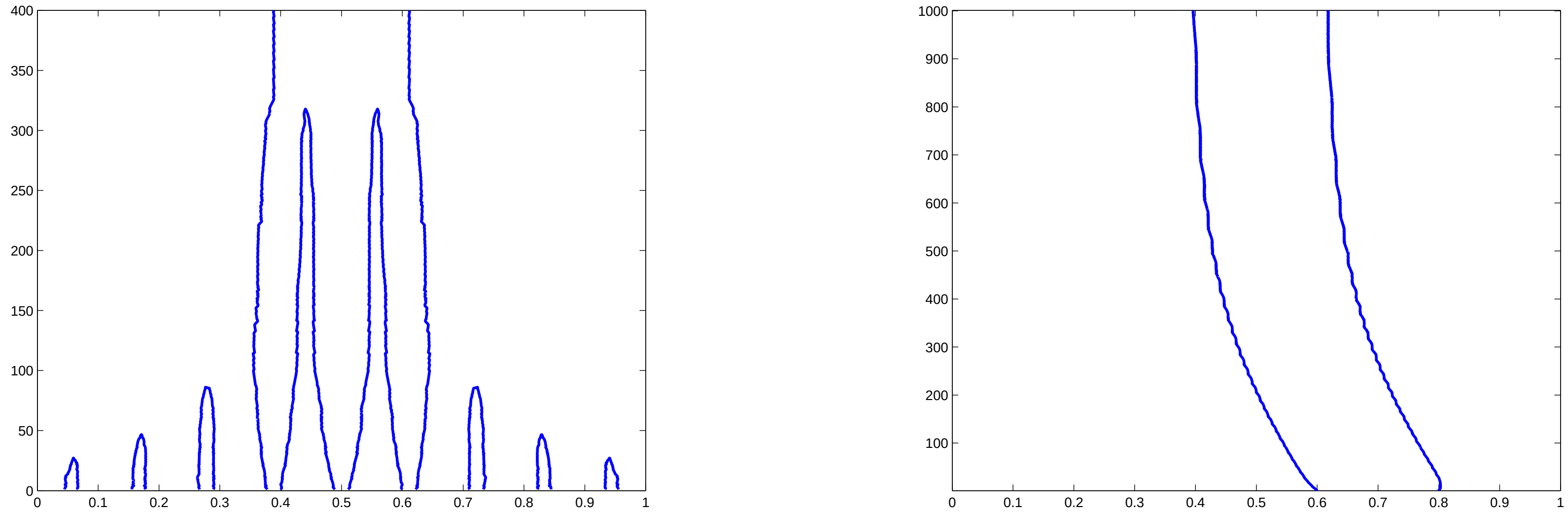


Figure 18: $(y_0, y_1) = (130 \sin(\pi x_1) \sin(\pi x_2/2), 0)$**,** $T = 2$**,** $a = 1.$ **- Evolution of the zero level set** $\{\boldsymbol{x} \in \partial\Omega \backslash \gamma_0, \psi_k(\boldsymbol{x})\} = 0$ **with respect to** k **-** $h = 1/151$**.**

Conclusions - Open Problems

A few remarks :

- Level set method is well suited and easy to carry out
- Gradient descent method works but leads to local minima
- First order gradient methods may be replaced by second order Newton type methods, less expansive in CPU time
- Numerical simulations leads to non-intuitive results. Even for symmetrical initial conditions and symmetric geometries, the optimal position is not necessary symmetrical and depends of a. This is due to over-damping phenomena and is generic for all systems

Extensions-

- Extend to the elasticity system with piezo device.
- Consider the case where the position of the dissipative zone ω depends on the time $t \in (0, T)$!
- Consider the exact controllability version : Find the optimal position of an exact control in order to minimize its norm.

$$min_\omega ||v_\omega||_{L^2(\omega \times (0,T))} \tag{44}$$

where v_ω is the HUM-control locate on ω.