"Anyonic statistics & Q.C." Jiannis K. Pachos

Conceptual Understanding + Analytic derivation + Physical Implementation

Bosons: trivial $: |\psi\rangle \to |\psi\rangle$ $\Leftarrow$ $|\psi\rangle \to |\psi\rangle$

Fermions: $: |\psi\rangle \to +|\psi\rangle$ $|\psi\rangle \to -|\psi\rangle$

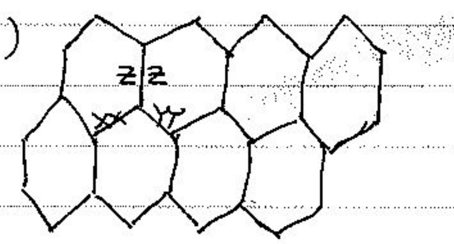
Abelian Anyons: $: |\psi\rangle \to e^{i\phi_{ab}}|\psi\rangle$ possible in 2-dim

(Aharonov-Bohm effect - Berry phases)

Use this for CP gate!

Model proposed by Kitaev: Honeycomb ~~lattice~~ spin-½ lattice

$$H = -J_x \sum_{x\text{-links}} \overbrace{X_j X_k}^{K_{jk}} - J_y \sum_{Y\text{-links}} \overbrace{Y_j Y_k}^{K_{jk}} - J_z \sum_{z\text{-links}} \overbrace{Z_j Z_k}^{K_{jk}} \quad (1)$$



$$= -\sum_{jk} J_{a_{jk}} \hat{K}_{jk}$$

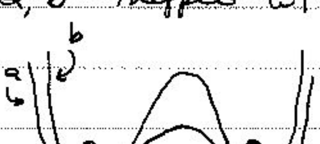
Physical implementation with optical lattices

Two atomic spieces a, b trapped with two opt. latt.

$|a\rangle|b\rangle$ basis

[J, U couplings], $J \ll U$

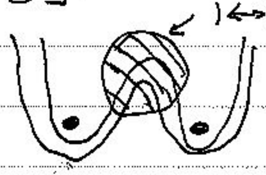
$\longrightarrow$



Effective interaction $H_{eff} = Z_1 Z_2$

(only phase accummulation no exchange of states)

Mott insulator 1 atom/site.



$H = \Omega\begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow E_\pm = \pm\Omega$ eigenvalues

$|\psi_\pm\rangle = (|a\rangle \pm |b\rangle)/\sqrt{2}$

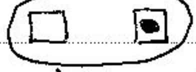
$|\psi_\pm\rangle$ basis:

$\Rightarrow H_{eff} = Z_1 Z_2\big|_{|\psi_\pm\rangle} = X_1 X_2\big|_{|a,b\rangle}$ basis

With superlattices it is possible to obtain the Ham. (1).

$J_\alpha \sim \frac{J^2}{U}$

Solve Hamiltonian by fermionisation

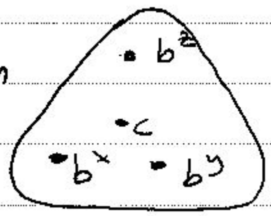
• Represent spins by fermions:  : Two fermionic modes a_1, a_1^+ $\quad a_2, a_2^+$

Space of states: $\left\{ \begin{matrix} |00\rangle & |01\rangle \\ |10\rangle & |11\rangle \end{matrix} \right\}$ Represent spin by $|\uparrow\rangle = |00\rangle$, $|\downarrow\rangle = |11\rangle$

Employ Majorana fermions ("Real" instead of "complex")

$$\left.\begin{matrix} c_0 = a_1 + a_1^+ = c & c_2 = a_2 + a_2^+ = b^y \\ c_1 = \dfrac{a_1 - a_1^+}{i} = b^x & c_3 = \dfrac{a_2 - a_2^+}{i} = b^z \end{matrix}\right\} \quad \begin{matrix} c_j^+ = c_j \\ c_j c_k + c_k c_j = 2\delta_{jk} \end{matrix}$$

Eigenstates of $D = b^x b^y b^z c$ with $d=+1$ are $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle = \alpha|\uparrow\rangle + b|\downarrow\rangle \in \mathcal{L}$

1-spin  $\sigma^\alpha \equiv i b^\alpha c$ for states in $\mathcal{L}$

$[\sigma^\alpha, D] = 0$, σ^α satisfy Pauli commun. rel.

$$\left[\begin{matrix} H\{\text{spin}\} = H\{ i b_j^\alpha c_j \} & : \text{Find eigenstates of } H \text{ which also} \\ [D, H] = 0 & \text{satisfy } D_j|\psi\rangle = |\psi\rangle \quad (\text{gauge invariance}) \end{matrix}\right]$$

Our case $\longrightarrow$ $\hat{H}_0 = -\sum_{(j,k)} J_{\underset{x,y,z}{\underset{\shortparallel}{\alpha(j,k)}}} \hat{K}_{jk} = -\sum_{(j,k)} J_\alpha (i\hat{b}_j^\alpha \hat{c}_j)(i\hat{b}_k^\alpha \hat{c}_k) = \dfrac{i}{2} \sum_{(j,k)} \hat{A}_{jk} \hat{c}_j \hat{c}_k$

$A_{jk} = 2 J_{\alpha(j,k)} \hat{U}_{jk}$ with $\hat{U}_{jk} = i \hat{b}_j^\alpha \hat{b}_k^\alpha$ $\left[\hat{U}_{kj} = -\hat{U}_{jk}, \ \hat{U}_{jk}^+ = \hat{U}_{jk}, \ (\hat{U}_{jk})^2 = 1 \right.$

• We have $[\hat{U}_{jk}, \hat{H}_0] = 0$. $\Rightarrow$ eigenvalues ± 1]
signs attached on each vertex.

$\Rightarrow \hat{U}_{jk}$: integrals of motion

.

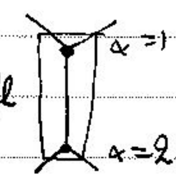
$\hat{U}_{jk}|\psi\rangle = u_{jk}|\psi\rangle$. If I fix u_{jk} at the beginning they remain constant. ($u_{jk} = 1$ for lowest energy)

Diagonalize $H\{ i b_j^\alpha c_j \}$ with $A_{jk} = 2 J_{\alpha(j,k)} u_{jk}$: real antisym.

$[\hat{D}, \hat{U}] \neq 0 \Rightarrow$ Symmetrize over "gauge transformations" i.e. Project onto the physical subspace.

The spectrum of H_0

$c_{j\alpha}$ j-cell



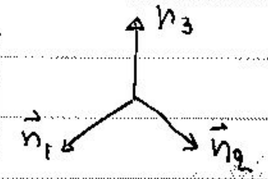
$$H = \sum_{j,\alpha,k,\beta} \frac{i}{4} A_{j\alpha,k\beta}\, c_{j\alpha} c_{k\beta}$$

$$\tilde{A}_{\alpha,\beta}(q) = \sum_{k} A_{0\alpha,k\beta}\, e^{i(\vec{q},\vec{r}_k)}$$

Momentum rep: $H = \sum_{q\alpha\beta} \frac{i}{4} \tilde{A}_{\alpha\beta}(q)\, \tilde{c}_\alpha(q) \tilde{c}_\beta(-q)$ with $\tilde{c}_\beta(-q) = \tilde{c}_\beta(q)^+$

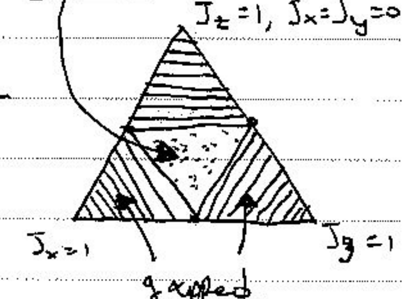
$$\tilde{A}(q) = \begin{pmatrix} 0 & f(q) \\ -f^*(q) & 0 \end{pmatrix} \Rightarrow \boxed{\varepsilon(q) = \pm |f(q)|}$$

with $\boxed{f(q) = J_x e^{i(\vec{n}_1,\vec{q})} + J_y e^{i(\vec{n}_2,\vec{q})} + J_z e^{i(\vec{n}_3,\vec{q})}}$



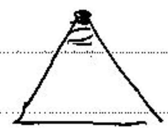
$\Rightarrow$ For $\left.\begin{array}{l} J_x \leqslant J_y + J_z \\ J_y \leqslant J_x + J_z \\ J_z \leqslant J_x + J_y \end{array}\right\}$ $\exists\, \vec{q} : \varepsilon(q) = 0. \Rightarrow$ gapless spectrum

The same phase on each triangle regions.



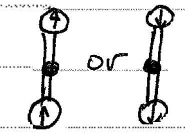
- gapped $\rightarrow$ excitations are abelian anyons
- gapless + $\vec{B}$ $\rightarrow$ gapped $\rightarrow$ excitations are non-abelian anyons

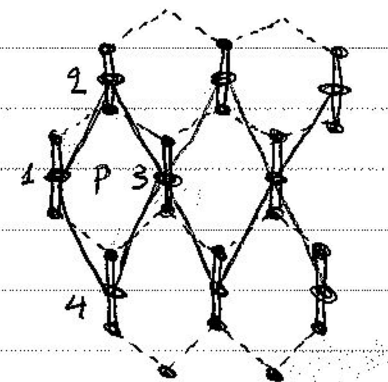
Abelian Anyons



Apply perturbation theory for $J_z \gg J_x, J_y$

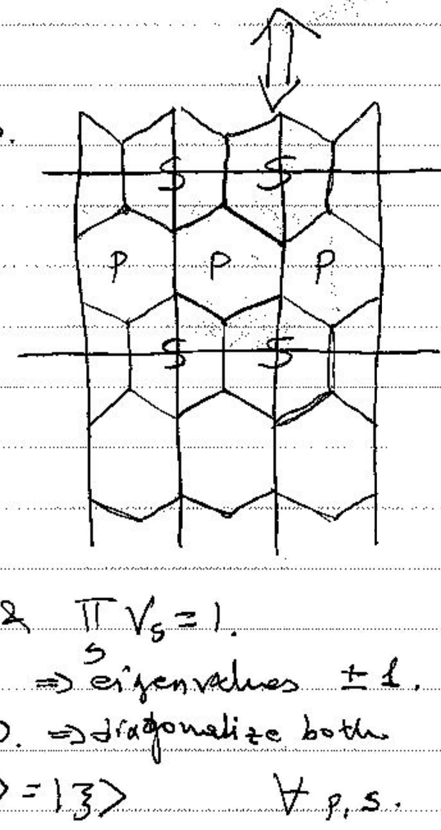
1^{st} order $J_x = J_y = 0 \Rightarrow H = -J_z \sum_{jk} z_j z_k$:

Eigenstates :  $\Rightarrow$ effective spins $\Uparrow$ or $\Downarrow$



4^{th} order $H_{eff} = -\frac{J_x^2 J_y^2}{16 J_z^3} \sum_p \underbrace{\sigma_{1p}^y \sigma_{2p}^z \sigma_{3p}^y \sigma_{4p}^z}_{\text{effective spins.}}$

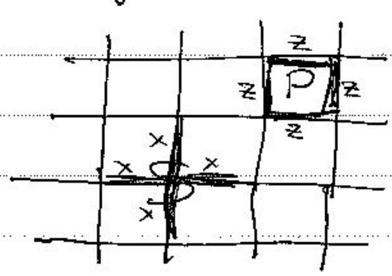
Rename $\sigma_j^y \longleftrightarrow \sigma_j^x$
on every other row of hexagons.



$\boxed{\text{Toric code}}$

$\Rightarrow H_{eff} = -J_{eff} \sum_{\text{plaquetes}} V_p - J_{eff} \sum_{\text{vertices}} V_s$

$V_p = \prod_{j \in bdry(p)} \sigma_j^z \qquad V_s = \prod_{j \in star(s)} \sigma_j^x$



! $\prod_p V_p = 1$ & $\prod_s V_s = 1$.
$V_p^2 = 1 = V_s^2 \Rightarrow$ eigenvalues ± 1.
$[V_p, V_s] = 0. \Rightarrow$ diagonalize both

Ground state $V_p|\xi\rangle = |\xi\rangle$ & $V_s|\xi\rangle = |\xi\rangle \quad \forall p, s.$

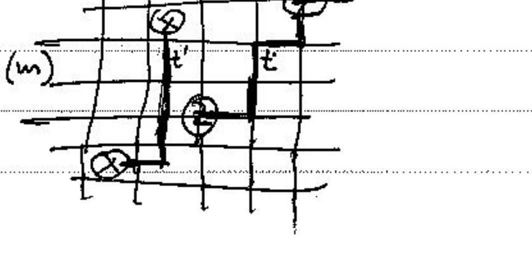
Excited states when or are violated

But as $\prod_p V_p = 1 = \prod_s V_s \Rightarrow$ elementary excitations (particles) are two particle states :

string operators

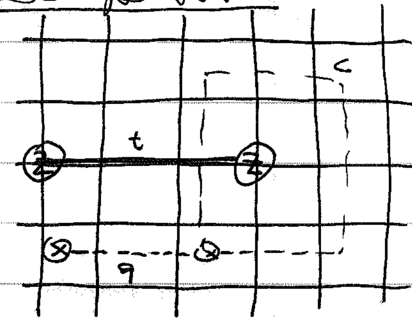
$S^z(t) = \prod_{j \in t} Z_j$, $|\psi^z(t)\rangle = S^z(t)|\xi\rangle$ (e)
(electric charges)

$S^x(t') = \prod_{j \in t'} X_j$, $|\psi^x(t')\rangle = S^x(t')|\xi\rangle$ (m)
(magnetic vortices)



$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

Abelian Anyon Statistics



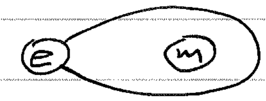
$|\Psi_{initial}\rangle = S^z(t)|\Psi^x(q)\rangle$

by circulation $\rightarrow$ $|\Psi_{final}\rangle = S^x(c)\, S^z(t)\, |\Psi^x(q)\rangle = -|\Psi_{initial}\rangle$

as $\{S^x(c), S^z(t)\} = 0$ anticommute

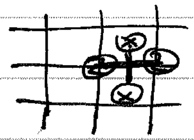
and $S^x(c)|\Psi^x(q)\rangle = |\Psi^x(q)\rangle$

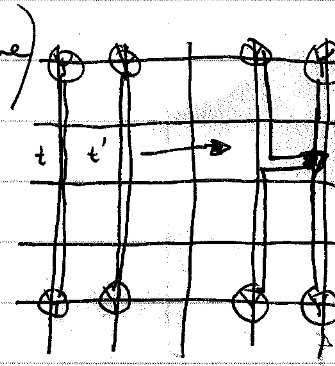
$\Rightarrow$ "$-$" sign $\Rightarrow$ abelian anyons.

$\Rightarrow$  $= -1$.

Fusion rules $e \times e = 1$, $m \times m = 1$

$e \times m = ?$

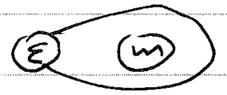
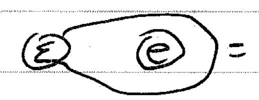
 $=$ $ZX = iY$. (what particles are there?)



$|\Psi_{initial}\rangle = S^Y(t)\, S^Y(t')\,|\xi\rangle$

by exchange $\rightarrow$ $|\Psi_{final}\rangle = (iY)^2|\Psi_{initial}\rangle = -|\Psi_{initial}\rangle$

$\Rightarrow$ Fermion (the one from diagonalisation)

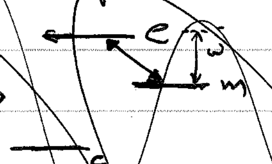
$\Rightarrow$ $\boxed{e \times m = \varepsilon}$ &  $= -1$  $= -1$.

$\varepsilon \times \varepsilon = 1$, $e \times \varepsilon = m$, $m \times \varepsilon = e$ similarly.

Quantum Computation

$|0\rangle_L \equiv |e,e\rangle$

$|1\rangle_L \equiv |m,m\rangle$

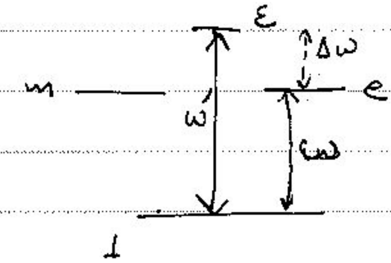


By resonant transition one can create $|e,e\rangle$ & $|m,m\rangle$ superpositions $|\Psi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L$

$\Rightarrow$ One qubit gates

$V(x,t) = (\underbrace{A \sin\omega t + B}_{\text{amplitude modulation}})\, \underbrace{\cos^2 kx}_{\text{lattice}}$

Quantum Computation



$|0\rangle_L \equiv |e,e\rangle$

$|1\rangle_L \equiv |m,m\rangle$

$$V(x,t) = \underbrace{(A \sin \omega t + B)}_{\substack{\text{amplitude}\\ \text{modulation}}} \underbrace{\cos^2 kx}_{\substack{\text{optical}\\ \text{lattice}}}$$

⇒ Can make coherent transitions between $|1\rangle, |\varepsilon\rangle, |m\rangle, |e\rangle$

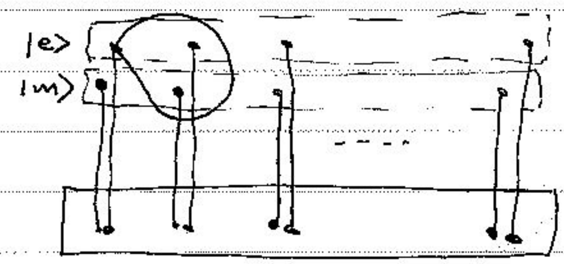
⇒ one qubit gates.

$|e\rangle|e\rangle \longrightarrow |e\rangle|e\rangle$

$|e\rangle|m\rangle \longrightarrow -|e\rangle|m\rangle$

$|m\rangle|e\rangle \longrightarrow |m\rangle|e\rangle$

$|m\rangle|m\rangle \longrightarrow |m\rangle|m\rangle$



⇒ two qubit CP gate.

Robust against perturbations: virtual creation of a ee (or other) pair ~~needs~~ happens with probability e^{-aL} where L is the distance between two encoding anyons.