3 Renormalization Group

- Running Coupling
- Asymptotic Freedom
- Running Mass
- Renormalization Group Equations
- Scaling
- Effective Field Theory
- Wilson Coefficients

EFFECTIVE (RUNNING) COUPLING

$$\alpha_R(\mu^2) \; = \; \alpha_0 \, \mu^{2\epsilon} \, \left\{ 1 + \frac{\alpha_0}{2\pi} \, \beta_1 \, \mu^{2\epsilon} \, \left[\frac{1}{\epsilon} + C' \right] + \cdots \right\}$$

Beta Function:

$$\mu \, \frac{d\alpha}{d\mu} \; \equiv \; \alpha \, \beta(\alpha) \qquad ; \qquad \beta(\alpha) \; = \; \beta_1 \, \frac{\alpha}{\pi} + \beta_2 \left(\frac{\alpha}{\pi} \right)^2 + \cdots$$

$$\ln\left(\frac{\mu}{\mu_0}\right) \; = \; \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \; = \; \int_{\alpha(\mu_0^2)}^{\alpha(\mu^2)} \frac{d\alpha}{\alpha\,\beta(\alpha)}$$

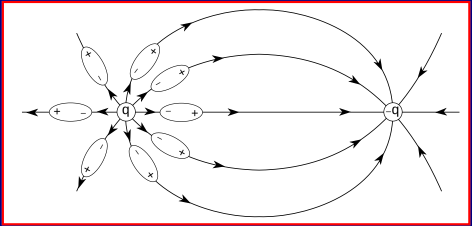
One Loop:

$$\alpha(Q^2) \; = \; \frac{\alpha(Q_0^2)}{1 - \beta_1 \, \frac{\alpha(Q_0^2)}{2\pi} \, \log\left(Q^2/Q_0^2\right)}$$

QED $\quad (N_f = 1)$:

$\beta_1^{\text{QED}} = \frac{2}{3}$ ⟹ $\alpha(Q^2)$ Increases with $Q^2 \equiv -q^2$

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \beta_1 \frac{\alpha(Q_0^2)}{2\pi} \log\left(Q^2/Q_0^2\right)}$$



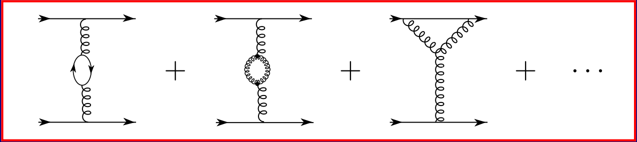
Virtual $f\,\bar{f}$ Pairs in **QED Vacuum**

Polarized Dielectric Medium

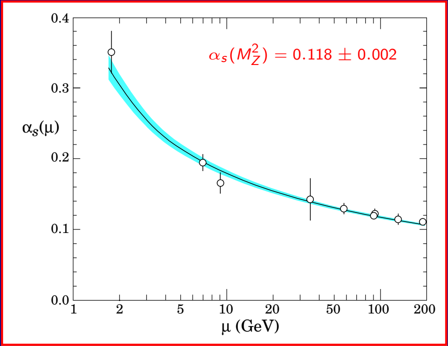
Charge SCREENING at Large Distances

$$\alpha = \alpha(m_e^2) = 1/137 \quad < \quad \alpha(M_Z^2) = 1/129$$

QCD:



$$\beta_1^{\text{QCD}} = \frac{2\,N_F - 11\,N_C}{6} < 0 \quad \Longrightarrow \quad \lim_{Q^2\to\infty} \alpha_s(Q^2) = 0$$



ASYMPTOTIC FREEDOM

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \beta_1 \dfrac{\alpha(Q_0^2)}{2\pi} \log\left(Q^2/Q_0^2\right)}$$

$$\alpha(Q^2) \; = \; \frac{\alpha(Q_0^2)}{1 - \beta_1 \, \frac{\alpha(Q_0^2)}{2\pi} \, \log\left(Q^2/Q_0^2\right)}$$

QED:

$$\beta_1^{\mathrm{QED}} = \frac{2}{3} \sum_f Q_f^2 N_f \; > \; 0 \qquad \Longrightarrow \qquad \lim_{Q^2\to 0} \alpha(Q^2) \; = \; 0$$

Quantum corrections make **QED irrelevant** at low energies

QCD:

$$\beta_1^{\mathrm{QCD}} = \frac{2\, N_F - 11\, N_C}{6} \; < \; 0 \qquad \Longrightarrow \qquad \lim_{Q^2\to 0} \alpha_s(Q^2) \; = \; \infty$$

Quantum corrections make **QCD relevant** at low energies

$\alpha(Q^2)$ Sums the Leading Logarithms

$$T(Q^2) \sim \alpha(\mu^2) \left\{ 1 + \frac{\beta_1}{2} \frac{\alpha(\mu^2)}{\pi} \ln\left(Q^2/\mu^2\right) + \cdots \right\}$$

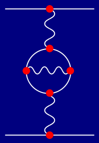
$$\sim \alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\beta_1}{2} \frac{\alpha(\mu^2)}{\pi} \ln\left(Q^2/\mu^2\right)}$$

$$= \alpha(\mu^2) \sum_{n=0} \left[\frac{\beta_1}{2} \frac{\alpha(\mu^2)}{\pi} \ln\left(Q^2/\mu^2\right) \right]^n$$

2 Loops (NLO)

$$\beta_2^{\mathrm{QED}} = \frac{N_F}{2} \quad ; \quad \beta_2^{\mathrm{QCD}} = -\frac{51}{4} + \frac{19}{12} N_F$$

$$(\beta_2^{\mathrm{QCD}} < 0 \ \text{ if } \ N_F \le 8)$$



$+ \quad \cdots$

$$\mu \frac{d\alpha}{d\mu} = \alpha\, \beta(\alpha) = \alpha \left[\beta_1 \frac{\alpha}{\pi} + \beta_2 \left(\frac{\alpha}{\pi}\right)^2 + \cdots \right]$$

$$\alpha(\mu^2) = \frac{\alpha(\mu_0^2)}{1 - \frac{\beta_1}{2} \frac{\alpha(\mu_0^2)}{\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right) - \frac{\beta_2}{2} \left(\frac{\alpha(\mu_0^2)}{\pi}\right)^2 \ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

$\alpha(Q^2)$ **sums** the leading $\left[\alpha(\mu_0^2)\, \ln\left(Q^2/\mu_0^2\right)\right]^n$ and

next-to-leading $\alpha(\mu_0^2) \left[\alpha(\mu_0^2)\, \ln\left(Q^2/\mu_0^2\right)\right]^n$ logarithms

RUNNING MASS

$$\mu \frac{dm}{d\mu} = -m \;\gamma(\alpha) = -m \left[\gamma_1 \frac{\alpha}{\pi} + \gamma_2 \left(\frac{\alpha}{\pi}\right)^2 + \cdots\right]$$

$$\ln\left(\frac{m}{m_0}\right) = \int_{m_0}^{m} \frac{dm}{m} = -\int_{\mu_0}^{\mu} \frac{d\mu}{\mu}\,\gamma(\alpha) = -\int_{\alpha(\mu_0^2)}^{\alpha(\mu^2)} \frac{d\alpha}{\alpha}\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

$$\frac{m(\mu^2)}{m(\mu_0^2)} = \left[\frac{\alpha(\mu^2)}{\alpha(\mu_0^2)}\right]^{-\gamma_1/\beta_1} \left\{1 + \frac{\beta_2}{\beta_1}\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right)\frac{\alpha(\mu^2) - \alpha(\mu_0^2)}{\pi} + \cdots\right\}$$

QED: $m = Z_2 Z_4^{-1} m_0 = \left[1 - \frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\,\frac{3}{\hat{\epsilon}}\right] m_0$ $\longrightarrow$ $\gamma_1^{\mathrm{QED}} = \frac{3}{2}$

$$\gamma_2^{\mathrm{QED}} = \frac{3}{16} - \frac{5}{12}\,N_F$$

m is a "renormalized" coupling

Depends on:

- The scale: $m(\mu^2)$
- The scheme: $\overline{\mathrm{MS}}$
- The QED/QCD Effective Field Theory
  (the number of active flavours N_F)

→ Matching Conditions

QCD:

Known to 4 loops: Chetyrkin, Vermaseren-Larin-Ritbergen

$\mathbf{N_F = 3}$:

$$\beta_1 = -4.5 \quad , \quad \beta_2 = -8 \quad , \quad \beta_3 = -20.12 \quad , \quad \beta_4 = -94.46$$

$$\gamma_1 = 2 \quad , \quad \gamma_2 = 7.6 \quad , \quad \gamma_3 = 24.84 \quad , \quad \gamma_4 = 88.53$$

Good perturbative convergence

$$\boxed{\frac{m_q(1\ \mathrm{GeV}^2)}{m_q(M_Z^2)} = 2.30 \pm 0.05}$$

POLE MASS

M $\equiv$ Pole of the renormalized propagator

$$M = \overline{m}\left\{1 + r_1\,\frac{\alpha(\overline{m})}{\pi} + r_2\left(\frac{\alpha(\overline{m})}{\pi}\right)^2 + \cdots\right\} \qquad ; \qquad \bar{m} \equiv m(m^2)$$

$$r_1^{\mathrm{QED}} = 1 \qquad ; \qquad r_1^{\mathrm{QCD}} = \frac{4}{3}$$

QED: Natural definition. **M $\equiv$ physical mass**

QCD:

- Confinement: Quarks are not asymptotic states
  Is there any physical quark pole ?
- **M** is defined perturbatively
- Bad convergence of perturbation theory

QCD MATCHING

$$(\mu > M) \qquad \mathcal{L}_{\rm QCD}^{(N_F)} \quad \Longleftrightarrow \quad \mathcal{L}_{\rm QCD}^{(N_F-1)} + \sum_{d_i>4} \frac{c_i}{M^{d_i-4}}\, O_i \qquad (\mu < M)$$

$$\alpha_s^{(N_F)}(\mu^2) \;=\; \alpha_s^{(N_F-1)}(\mu^2)\, \left\{ 1 + \sum_{k=1}^{\infty} C_k(L)\, \left[\frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi}\right]^k \right\}$$

$$L \equiv \ln\left(\mu^2/m_q^2\right)$$

$$m_q^{(N_F)}(\mu^2) \;=\; m_q^{(N_F-1)}(\mu^2)\, \left\{ 1 + \sum_{k=1}^{\infty} H_k(L)\, \left[\frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi}\right]^k \right\}$$

- Matching conditions known to 3 loops: $C_{1,2,3}\;,\; H_{1,2,3}$
  (Chetyrkin et al, Larin et al)
- L dependence known to 4 loops: $C_4(L)\;,\; H_4(L)$
- $\alpha_s(\mu^2)$ is not continuous at threshold

RENORMALIZATION GROUP EQUATION

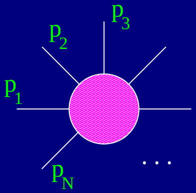
R dimensionless observable depending on a single scale **Q** $(m = 0)$

- Naive scaling: $R(Q^2) = \text{constant}$
- Renormalizable QFT: $R = R[Q^2/\mu^2, \alpha(\mu^2)]$

R does not depend on μ:

$$\mu \frac{dR}{d\mu} = 0 = \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \alpha}{\partial \mu} \frac{\partial}{\partial \alpha}\right) R = \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha)\, \alpha \frac{\partial}{\partial \alpha}\right) R$$

i.e. $$R[Q^2/\mu^2, \alpha(\mu^2)] = R[1, \alpha(Q^2)]$$



$$\Gamma_0(p_i;\alpha_0,m_0;\epsilon) \;=\; Z_\Gamma(\epsilon,\mu)\,\Gamma(p_i;\alpha,m;\mu)$$

$$\left(\mu\,\frac{d}{d\mu}+\gamma_\Gamma(\alpha)\right)\,\Gamma \;=\; 0 \qquad ; \qquad \gamma_\Gamma(\alpha)\;\equiv\;\frac{\mu}{Z_\Gamma}\,\frac{dZ_\Gamma}{d\mu}$$

$$\left(\mu\,\frac{\partial}{\partial\mu}+\beta(\alpha)\,\alpha\,\frac{\partial}{\partial\alpha}-\gamma(\alpha)\,m\,\frac{\partial}{\partial m}+\gamma_\Gamma(\alpha)\right)\,\Gamma \;=\; 0$$

$$\Gamma[p_i;\alpha(\mu^2),m(\mu^2);\mu] \;=\; \Gamma[p_i;\alpha(\mu_0^2),m(\mu_0^2);\mu_0]\;\exp\left\{-\int_{\alpha(\mu_0^2)}^{\alpha(\mu^2)}\frac{d\alpha}{\alpha}\,\frac{\gamma_\Gamma(\alpha)}{\beta(\alpha)}\right\}$$

SCALING: $\quad \Gamma(\xi\, p_i; \alpha, m; \mu) \;=\; \xi^{d_\Gamma}\, \Gamma(p_i; \alpha, m/\xi; \mu/\xi)$

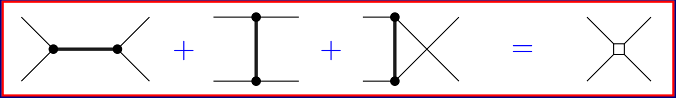
$$\left\{ \xi \frac{\partial}{\partial \xi} - \beta(\alpha)\, \alpha \frac{\partial}{\partial \alpha} + [1 + \gamma(\alpha)]\; m \frac{\partial}{\partial m} - \Big[d_\Gamma + \gamma_\Gamma(\alpha) \Big] \right\} \Gamma(\xi\, p_i; \alpha, m; \mu) \;=\; 0$$

$$\Gamma[\xi\, p_i; \alpha(\mu^2), m(\mu^2); \mu] \;=\; \xi^{d_\Gamma}\; \exp\left\{ \int_{\alpha(\mu^2)}^{\alpha(\xi^2\mu^2)} \frac{d\alpha}{\alpha}\, \frac{\gamma_\Gamma(\alpha)}{\beta(\alpha)} \right\}\; \Gamma[p_i; \alpha(\xi^2\mu^2), m(\xi^2\mu^2)/\xi; \mu]$$

EFFECTIVE FIELD THEORY

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi$$

$$\sigma(\phi\phi\to\phi\phi) \;\sim\; \frac{1}{E^2} \;\times\; \begin{cases} (\lambda/E)^4 \;, & (m \ll M \ll E) \\ (\lambda/M)^4 \;, & (m, E \ll M) \end{cases}$$



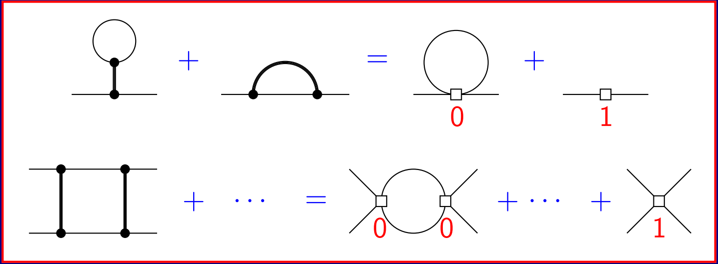
$\mathbf{E, m \ll M}$

$$\frac{\lambda^2}{s-M^2} = -\frac{\lambda^2}{M^2}\sum_{n=0}\frac{s}{M^2} \quad \Longrightarrow \quad \mathcal{L}_{\mathrm{eff}}(\phi) = \sum_i c_i\, O_i(\phi)$$

$$[O_i] = d_i \quad ; \quad c_i \sim \frac{\lambda^2}{M^2}\,\frac{1}{M^{d_i-4}}$$

One–Loop:

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\, a\,(\partial\phi)^2 - \frac{1}{2}\, b\,\phi^2 + c\,\frac{\lambda^2}{8M^2}\,\phi^4 + \cdots$$



$$a = 1 + a_1\,\frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad b = m^2 + b_1\,\frac{\lambda^2}{16\pi^2} + \cdots$$

$$c = 1 + c_1\,\frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad \cdots$$

Wilson Coefficients:

$$\mathcal{L} = \sum_i \frac{c_i}{\Lambda^{d_i-4}}\, O_i$$

$$\langle O_i \rangle_B = Z_i(\epsilon,\mu)\, \langle O_i(\mu)\rangle_R \qquad ; \qquad \mu\,\frac{d}{d\mu}\,\langle O_i\rangle_B = 0$$

$$\left(\mu\,\frac{d}{d\mu} + \gamma_{O_i}\right)\langle O_i\rangle_R = 0 \qquad ; \qquad \gamma_{O_i} \equiv \frac{\mu}{Z_i}\,\frac{dZ_i}{d\mu} = \gamma_{O_i}^{(1)}\,\frac{\alpha}{\pi} + \gamma_{O_i}^{(2)}\left(\frac{\alpha}{\pi}\right)^2 + \cdots$$

$$\mu\,\frac{d}{d\mu}\, c_i(\mu)\,\langle O_i\rangle_R = 0 \quad \longrightarrow \quad \left(\mu\,\frac{d}{d\mu} - \gamma_{O_i}\right) c_i = 0$$

$$\longrightarrow \quad \begin{aligned} c_i(\mu) &= c_i(\mu_0)\,\exp\left\{\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha}\,\frac{\gamma_{O_i}(\alpha)}{\beta(\alpha)}\right\} \\ &= c_i(\mu_0)\left[\frac{\alpha(\mu^2)}{\alpha(\mu_0^2)}\right]^{\gamma_{O_i}^{(1)}/\beta_1}\left\{1 + \cdots\right\} \end{aligned}$$